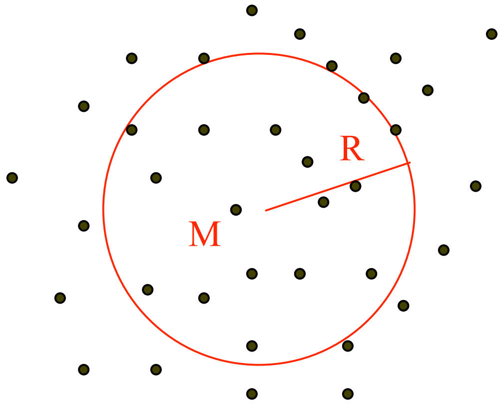
Newtonian Kinematics of the Universe

Let’s follow the evolution of a sphere of radius R, containing a mass M, expanding according to Hubble’s Law.



The mass inside the sphere is constant as it expands. Let’s see what happens to the energy. Conservation of energy requires that KE + PE = a constant, which we’ll call –k/2.

Kinetic energy per unit mass, KE = $v^2/2$
Potential energy per unit mass, PE = -GM/R

So: $v^2/2 - GM/R = -k/2.$

Hubble’s law says: $v = H \times R$, where H = the Hubble constant.

And mass is volume times density: $M = \frac{4}{3}\pi R^3 \rho$, where ρ is the density.

This gives:

$$\frac{1}{2}H^2R^2 - \frac{4}{3}\pi G R^2 \rho = -\frac{1}{2}k$$

Now divide both sides by $\frac{1}{2}R^2$ to get: $H^2 - \frac{8}{3}\pi G \rho = -\frac{kc^2}{R^2}$

This is the **Friedmann Equation**, which describes the *standard model*, one in which the cosmological constant, Λ, equals 0. The c^2 on the right-hand side comes from the formal relativistic derivation.

Note that ρ and R are not independent, since the quantity ρR^3 is constant (conservation of mass).

We get three cases depending on the value of **k**:

A. $H^2 > \frac{8}{3}\pi G\rho$ for **k** <0

B. $H^2 < \frac{8}{3}\pi G\rho$ for **k** > 0

C. $H^2 = \frac{8}{3}\pi G\rho$ for **k** $= 0$

A corresponds to an **open** universe which will expand forever. It has negative curvature (saddle-surface analogy).

B corresponds to a **closed** universe, which will eventually recollapse. It has positive curvature (spherical analogy).

C corresponds to the **boundary** case, where the universe will just barely expand forever (Euclidean plane analogy). In this case, we can calculate the value of the density required.

Critical density: $\rho=\rho_{\text{critical}}=\dfrac{3H^2}{8\pi G}$

Note that ρ_{critical}, the critical density, is a function of time, since H is a function of time.