A perturbation map of planetary rings

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Introduction

Following Frøyland (Frøyland, Jan. *Introduction to Chaos and Coherence*, Institute of Physics Publishing, New York, 1992), I describe a map of particle radius and angle in a planetary ring under the perturbing influence of a satellite in a higher orbit. Saturn's rings and it's satellite Mimas are taken as a paradigmatic system.

Mimas orbits at a mean distance of $\sigma = 185.7 \times 10^3$ km from the center of Saturn, which has a radius $R_s = 60.4 \times 10^3$ km.

Model

Kepler's third law states that the square of the orbital period is proportional to the cube of the semi-major axis of the orbit, $T^2 \propto a^3$. For Mimas, $T_m \propto \sigma^{3/2}$. For a ring particle with an orbit of radius r_n, $T_n \propto r_n^{3/2}$. After a time T_m, Mimas will have covered an angle 2π, while the ring particle will have covered an angle $\frac{2\pi}{T_n}T_m = 2\pi\left(\frac{\sigma}{r_n}\right)^{3/2}$. If the initial relative angle between Mimas and the particle is θ_n, then after one period the new relative angle will be

$$\theta_{n+1} = \theta_n + 2\pi\left(\frac{\sigma}{r_n}\right)^{3/2}$$

.

We can approximate the radial acceleration of a ring particle with the equation

$$\frac{\Delta v_r}{\Delta t} = \frac{v_r(t+\Delta t) - v_r(t)}{\Delta t} = \frac{r(t+\Delta t) - r(t)}{\Delta t^2} - \frac{r(t) - r(t-\Delta t)}{\Delta t^2}$$

. We can solve this equation for $r(t+\Delta t)$ and write

$$r_{n+1} = 2r_n - r_{n-1} - \Delta v_r \Delta t$$

, where we have exchanged the functional dependence on time for discreet subscripts. We want to take some average over a period for $\Delta v_r \Delta t$. Let us write the equation, rather arbitrarily, as

$$r_{n+1} = 2r_n - r_{n-1} - a\frac{\cos(\theta_n)}{(r_n - \sigma)^2}$$

, where a has units km^3. The $\cos(\theta_n)$ is there for periodic angular dependence, the $1/(r_n - \sigma)^2$ comes from the inverse-square gravitational force. a should be proportional to the mass of the perturbing body. Notice it has the same units as $(G\ M\ \Delta t^2)$, where M is the mass.

Now we have a map that can be programmed into a computer to explore the stability of particle orbits at varying radii.

See http://www.sccs.swarthmore.edu/ roban/rings/rings.html for an application of this map.