RATIONAL POLARIZATION:

Asymptotic belief divergence in response to common information

Isaac Loh* Gregory Phelan†

This version: December 16, 2016

Abstract

When information is of lower dimension than the model generating the data, Bayesian learning need not converge to the truth. Because the information is of lower dimension than the model, agents face an identification problem, affecting the role of data in inference. We provide conditions under which Bayesian learning is asymptotically inconsistent with positive probability, and sometimes almost surely. Robustly, two agents with differing priors who observe identical, unambiguous data may disagree forever, with stronger disagreement the more data is observed. Agents rationally use common observations to differentially update beliefs about different parameters.

Keywords: beliefs, polarization, learning, Bayesian updating

JEL classification: D10, D80

1 Introduction

People disagree, sometimes forever. Standard models of Bayesian learning suggest that when presented with the same evidence, agents should update their beliefs "*toward* that evidence" so

*Northwestern University. Email: IsaacLoh2015@u.northwestern.edu .

†Department of Economics, Williams College. Email: gp4@williams.edu.

Website: `https://sites.google.com/site/gregoryphelan/`.

We are grateful for feedback from Matthew Chao, Asim Khwaja, Rachel Kranton, Bruce Sacerdote, Larry Samuelson, and Arunava Sen.

that agents' beliefs are more similar. Even if agents continue to disagree, it is supposed, they should disagree less, or differently. However, in reality beliefs often diverge and people become polarized.[1] Surprisingly, polarization occurs even when people face the same "evidence," and especially if they see lots of evidence. People constantly interpret the same data as evidence in favor of their prior beliefs. In particular, people may use the same evidence to draw inference about different variables. Consider two examples:

(i) Two economists, a Keynesian and a Neoclassical, walk into a bar. There was a recent stimulus package, and the new GDP results are sluggish. The Neoclassical says, "Goes to show that stimulus doesn't work." The Keynesian replies, "Oh no, goes to show that the economy is much worse than we thought." (Perhaps later adding that the stimulus was poorly designed.) Meanwhile, in another bar in another country far away, a stimulus bill has passed with apparently great results. "I guess the economy was already out of the recession," says the Neoclassical.

(ii) A liberal and a conservative are watching the news together. After a string of media coverage providing evidence that the country has become more liberal, the liberal says, "The country is coming my way." The conservative, responds, "No, more evidence of the media's liberal bias."

In each of these cases, the exact same evidence is interpreted in completely different ways, even confirming the prior that each person had before. But the observers are not rejecting the data, nor are the data ambiguous and unclear: they are simply using the evidence in different, rational ways. What these examples illustrate is that the world is *multi-dimensional*—there are a number of factors that contribute to what we see—but the data we see is nearly always of lower dimensionality than the world. In other words, we live with identification problems. When our data are of lower dimensionality than that of the generating process, some observations do not

[1]There is ample empirical evidence documenting belief divergence in many environments regarding different issues: religion (Batson, 1975); death penalty (Lord et al., 1979); nuclear power (Plous, 1991); caffeine (Chaiken et al., 1992); sexuality (Munro and Ditto, 1997); affirmative action and gun violence (Taber and Lodge, 2006). Darley and Gross (1983) provide evidence of people interpreting the same evidence differently in light of earlier evidence, and Hirshleifer and Teoh (2003) document how the presentation of accounting information affects its interpretation.

identify the underlying parameters. For the economists at the bar, GDP is a function of the state of the economy, the effectiveness of stimulus, and potentially how well designed that stimulus was. For the liberal and the conservative, the politics of news reflect the underlying beliefs of the population and how accurately those are reported. Each observer uses the data—they do not ignore the observation or assume it is just "noise"—but they rationally use the data to make inference about different underlying variables. The Keynesian infers that weak GDP data means something about the economy, not the effectiveness of stimulus, while the Neoclassical infers weak GDP means the fiscal multiplier is low.[2]

What is most interesting is not that people can disagree after a single observation, but that with *many* observations people may continue to disagree, and disagree further, even becoming completely convinced of their beliefs. The striking regularity is that polarization persists or increases when people observe many identical data points. Polarization can occur even if many observations are definitively revealing about the state of the world, so long as enough observations can rationally be used to make inference about different underlying variables.[3]

In this paper we show that when world is of higher dimensionality than the data people receive, (i) rationally updated posterior beliefs will converge, though not necessarily to the true values, and as a result (ii) heterogeneous priors may *rationally* diverge when agents' initial beliefs are polarized. In particular, these results are likely to occur when unidentified observations occur frequently: beliefs are likely to converge to values confirming initial beliefs, whether those beliefs are correct or not. In other words, posterior beliefs converge to confirm *relative* initial beliefs: what matters is which beliefs are held more strongly, not the absolute levels of beliefs.

To show these results, we consider a simple model in which there are two state variables, θ and σ, that determine whether realization variables t and s will be successes or failures. What

[2]We show that updating beliefs in this way may be a completely rational approach in light of identification issues. We do not make behavioral assumptions or suppose any bounded rationality. Of course, behavioral biases may provide other reasons why people interpret data in favor of their priors. That said, it is well-known that is easier to convince oneself of a lie that is largely based in fact, so the prevalence of behavioral biases may reflect the truth that a multidimensional world makes it quite possible to believably interpret data in favor of priors.

[3]A classic example of people observing exactly the same evidence and drawing polarized conclusions is the O.J. Simpson trial, in which the same evidence was interpreted vastly differently (see Uschel, 1995). Our paper is about how *repeated* observations can lead to polarization. Thus, an example would be that different people interpret multiple cases of police shootings in ways that lead to increased polarization (for evidence see Pew Research Center, 2014).

is observed is not t and s but the sum: agents see the number of successes but not the composition. Hence, observations of "two" or "zero" successes are clearly identified, while observations of "one" are "unidentified." Agents hold beliefs about the states θ and σ, which can be good or bad, and update their beliefs about the states in light of observations. Critically, when interpreting "unidentified observations" (ones), the likelihood function for one state (say θ) depends on beliefs about the other state (say σ). Technically, when agents have different beliefs, they have different likelihood functions. Thus, agents update beliefs differently when observing unidentified observations. Likelihood functions diverge more if beliefs are dissimilar. Thus, differing priors lead to divergent likelihood functions and divergent posteriors.

When unidentified observations are relatively unlikely, beliefs converge to the truth almost surely for any priors. However, when unidentified observations are relatively likely, then robustly with positive probability beliefs converge, though not necessarily to the truth. Instead, if a person has a prior that it is *more* likely that the good aggregate state is θ, then asymptotically with positive probability the person will believe *with certainty* that θ is the good state and that σ is the bad state (vice versa if the prior is that σ is more likely to be good). Thus, if two people disagree about which state is more likely to be good—regardless of the overall level of beliefs about each state—then robustly beliefs diverge asymptotically with positive probability. Furthermore, under certain conditions beliefs diverge asymptotically with probability one. In these cases, more data leads to greater polarization rather than greater agreement.

We theoretically characterize the limiting properties of beliefs in our model and show that there are robust conditions under which beliefs converge to the wrong values, leading to asymptotic polarization. In addition, we provide simulation results illustrating that it is almost always the case that polarization can occur with positive probability when unidentified observations are relatively likely. Thus, high-dimensionality may lead to asymptotic extreme polarization when unidentified observations are likely, but will lead to asymptotic agreement when identified observations are relatively likely. Or put differently, when unidentified observations are likely, agents in an economy with *common* priors may have collective beliefs converge to something other than the truth.

Related Literature Several papers consider how observing a small number of signals can lead to belief polarization. Benoît and Dubra (2014) show that as a result of identification problems of the type we consider, divergence occurs for intermediate values of information, but not for extreme values of information (which are more informative of the underlying identification structure).[4] Similarly, Baliga et al. (2013) show that polarization can occur as an optimal response to ambiguity aversion, a possibility only at signal with an intermediate likelihood ratio. Andreoni and Mylovanov (2012) also consider polarization about optimal actions when agents receive two-dimensional information to form a one-dimensional opinion. They consider that agents receive information about only one of two states, rather than information that is a function of the two states, and must make inference on what action is preferred, in light of beliefs about the other state (disagreement persists because agents discount information filtered through the actions of others).

We show that, in fact, even with infinitely many signals, beliefs may diverge, even when many of those signals are in fact extreme. In fact, it is possible that disagreement grows with the number of realizations, diverging to completely polarized views of the world. Like the intermediate results of Benoît and Dubra (2014) and Baliga et al. (2013), our asymptotic result holds when the true state of the world is in an "intermediate region", which will lead toward a higher frequency of unidentified signals. Thus, disagreement persists asymptotically only when the fundamentals are not extreme, leading to many unidentified observations.

Our asymptotic result differs from Acemoglu et al. (2016), who show that *agreement* does not necessarily follow even when learning does. They consider a model in which agents have different priors about the state, *and* they also have different beliefs about how the state maps to signals; in other words, they disagree about how much noise there is. As a result, agents also face an identification problem regarding how to interpret signals. They show that even though agents learn the true state asymptotically, this does not mean that agents will *agree* asymptotically because the likelihood ratios of their beliefs need not converge. In contrast, in our model agents need not learn the truth asymptotically, and disagreement can be exacerbated by the same information. The

[4]Similar observations are made by Dixit and Weibull (2007) and Jern et al. (2014). Kondor (2012) shows that public information can increase disagreement when traders have private information and different trading horizons (they update higher-order expectations about beliefs of others).

results differ because the nature of the identification problems we consider are different.[5]

Fryer et al. (2015) consider a model in which agents may receive "ambiguous" signals, which are interpreted in light of current priors (as in our model) and "stored" as an *un*ambiguous signal. Their convergence result is driven by bounded memory, where agents do not retain whether signals were ambiguous or not, but remember the interpreted value. Crucially, if ambiguous signals could be stored as such, learning (and agreement) would occur. Their setup shares many features with ours; however, our paper is fundamentally about the difference between the *dimensionality* of observations and the generating parameters, not about ambiguity. In our model, a signal of one is "ambiguous" only in the sense that the underlying values cannot be *identified*, and the way that agents update their posteriors depends on their beliefs—but agents *unambiguously know* that a one is a one and store the observation as a one; the observation is perfectly clear, but how it is interpreted differs because of the identification problem.

Our treatment of dimensionality in learning is applicable to the broad class of inference problems in economics in which agents are tasked with learning the distribution of a multidimensional x by observing oscillations in the one-dimensional function (or functional) $g(x)$ (this was the case in the two examples discussed earlier in the introduction). In our model, agents are completely rational, but there is a fundamental limitation, inherent to the model but not agents, in the ability of information to identify the model.[6]

Many papers provide theories (often with a behavioral assumption) of polarization and convergence to incorrect beliefs. Rabin and Schrag (1999) consider confirmatory bias; Eyster and Rabin (2010) show how "social confirmation bias" leads to herding with positive probability to incorrect actions; Ortoleva and Snowberg (2015) consider the role of overconfidence in political behavior to explain ideological extremeness; Schwartzstein (2014) shows that selective attention to informa-

[5]Nonetheless, our model can also deliver the result of asymptotic disagreement even when learning occurs. See Proposition 1 and Lemma 4.

[6]In the model of Fryer et al. (2015), agents interpret observations differently because of bounded rationality and because ambiguous signals are sufficiently likely. Therefore, divergence occurs when beliefs about the states are greater than or less than one-half and ambiguous signals occur with a sufficiently high frequency. In our model, divergence can occur when beliefs about one state are *greater* than the other, regardless of how high each belief is, whenever unidentified observations occur relatively frequently. Thus, divergence occurs when *relative* beliefs differ and the identification problem is sufficiently severe. Furthermore, as we discuss below, in our model, both agents will converge to the truth when the fundamentals in the model are not polarized (states are either (1,1) or (0,0)), whereas polarization occurs when fundamentals are, in a sense, polarized (states are (1,0) or (0,1)).

tion can lead to persistently incorrect beliefs (see also Sundaresan and Turban, 2014). Heidhues et al. (2015) consider a model in which agents are (overly) optimistic about their ability, choose an effort level, and observe the outcome. Beliefs about fundamentals diverge from reality precisely because agents change their effort level in a self-defeating, misguided way. Thus, incorrect learning follows because agents endogenously observe different signals based on their effort. In our model agents observe exactly the same signals and yet learn differently.[7]

A Simple Example

We first provide a simple example to illustrate how Bayesian learning is affected by the identification problem. In the next section we present the full model and characterize asymptotic properties.

Let there be two underlying state variables t and s that jointly determine the aggregate realization $y(t,s)$. The two state variables can each take two values in $\{0,1\}$. The aggregate realization is defined by the following function:

$$y(t,s) = t + s. \tag{1}$$

In other words, a realization of $y = 2$ or $y = 0$ definitively says the value of the underlying variables, but a realization of 1 yields an identification problem: there are two possible sets of parameters that would provide that realization.[8]

Let there be two agents denoted $i = 1,2$. Agent i holds beliefs $\Pr_i(t = 1) = P_i$ and $\Pr_i(s = 1) = Q_i$. Agents are rational: when they see the realization y they update their beliefs P_i and Q_i according to Bayes' Rule. Denote posterior beliefs by $\hat{P}_i$ and $\hat{Q}_i$. Consider two agents faced with a moderate realization $y = 1$ trying to learn the effectiveness of the stimulus. From Bayes' Rule the posteriors, given observation $y = 1$, are given by

$$\hat{P} = \frac{P(1-Q)}{P(1-Q)+(1-P)Q}, \quad \hat{Q} = \frac{Q(1-P)}{P(1-Q)+(1-P)Q}. \tag{2}$$

[7] See Eyster et al. (2014) for a model in which learning follows by observing the actions of others, which are endogenously determined as a function of expected payoffs.

[8] For example, one could interpret t as the strength of the economy and s as the effectiveness of a stimulus package. A strong economy and an effective stimulus lead to high output (similarly for weak variables), but a mix of strong economy and ineffective stimulus, or the reverse, will lead to a moderate realization.

Crucially, the posterior for P depends on beliefs about Q (vice versa).[9] How might beliefs diverge in light of information? We say that agents have strongly polarized beliefs if $P_1 > \frac{1}{2} > P_2$ and $Q_2 > \frac{1}{2} > Q_1$. Thus, agent 1 is optimistic about t but not s, and agent 2 is optimistic about s and not t. Divergence occurs when beliefs are strongly polarized. However, beliefs can diverge relative to each other without agents becoming strongly polarized. We say that polarization increases *relatively* if $\frac{\hat{P}_1}{\hat{P}_2} > \frac{P_1}{P_2}$ and $\frac{\hat{Q}_2}{\hat{Q}_1} > \frac{Q_2}{Q_1}$.

Lemma 1. *Suppose $P_1 > \frac{1}{2} > P_2$ and $Q_2 > \frac{1}{2} > Q_1$. Then polarization increases and posteriors diverge further if $y = 1$: $\hat{P}_1 > P_1 > P_2 > \hat{P}_2$ and $\hat{Q}_2 > Q_2 > Q_1 > \hat{Q}_1$. Relative polarization increases if*

$$\frac{\Pr_1(s=0)}{\Pr_2(s=0)} > \frac{\Pr_1(y=1)}{\Pr_2(y=1)}, \qquad \frac{\Pr_2(t=0)}{\Pr_1(t=0)} > \frac{\Pr_2(y=1)}{\Pr_1(y=1)}. \tag{3}$$

The results follow immediately from the equations for posteriors. When beliefs are polarized in this sense, the same information leads to increased polarization. Agent 1 becomes relatively more optimistic about t and less optimistic about s, and the reverse is true for agent 2. Thus, agents draw divergent conclusions from the same data; the observation works as evidence in favor of greater polarization.

2 The Model

In the static setup a realization of $y = 0$ or $y = 2$—evidence that cannot be interpreted differently—would cause beliefs to converge, and such a realization would cause both agents to admit that they were wrong about something. Thus, one might wonder if given enough observations beliefs must eventually converge. We now extend the model to answer precisely this question and show that, when observations of $y = 1$ are relatively likely, more observations increase the likelihood of polarization. We characterize the limiting behavior of posterior beliefs with theoretical and simulation results. Proofs not in the text are given in the appendix.

[9] In fact, the likelihood ratio for the odds ratio for P is $\frac{1-Q}{Q}$ and for Q is $\frac{1-P}{P}$, which is to say the likelihood ratio is completely determined by beliefs about the other varibles.

Setup Suppose now that there are two aggregate states, θ and σ, which determine the frequency of t and s respectively, and the aggregate states take binary values (think "good" or "bad"). In particular, t-successes occur with higher probability when $\theta = 1$ (θ is "good"), and s-successes occur with higher probability when $\sigma = 1$. Thus,

$$\Pr(t = 1|\theta = 1) = p_H > p_L = \Pr(t = 1|\theta = 0), \text{ and} \tag{4}$$

$$\Pr(s = 1|\sigma = 1) = q_H > q_L = \Pr(s = 1|\sigma = 0), \tag{5}$$

so that $t = 1$ is more likely when $\theta = 1$ and $s = 1$ is more likely when $\sigma = 1$. Let all probabilities p, q lie strictly between zero and one. The realization function is $y(t, s) = t + s$, as before. Thus, the aggregate states θ and σ determine the frequency of observations $y \in \{0, 1, 2\}$. The mapping from aggregate states θ and σ to t and s is common knowledge: agents only disagree about the likelihood of θ and σ, but not about how those states translate into realizations y.[10]

Belief Updating Let an agent hold beliefs P_n and Q_n about θ and σ (the probability the states are good), where n denotes the number of signals observed (also the period). Agents observe signal y in period $n+1$ and use Bayes' Rule to update their beliefs about the two aggregate states to P_{n+1} and Q_{n+1}. Thus, agents update beliefs sequentially, using only the current prior and the current information, together with Bayes' Rule, to form their posteriors.[11]

Define $\overline{p} = Pp_H + (1-P)p_L$ and $\overline{q} = Qq_H + (1-Q)q_L$ to be the ex-ante expected realizations of t and s given beliefs P, Q (notice we will suppress the sub-n subscripts when all variables share the same n value). It is convenient to work with odds ratios. Define $O^P = \frac{P}{1-P}$, and $O^Q = \frac{Q}{1-Q}$. Let

[10] Contrast this assumption with the information structure in Acemoglu et al. (2016), where a one-dimensional state θ produces signals with probability p_θ, and agents potentially disagree about p_θ as well as the probability of θ.

[11] Since agents update sequentially, agents are not strictly Bayesian in the classical sense of Doob in which agents obtain belief P_{n+1} by updating a fixed *initial prior* P_0 in conjunction with all $n+1$ observations at once, rather than sequentially. Since the learning process is path dependent (Lemma 3), sequential updating matters for the results.

$O^P(y), O^Q(y)$ denote the updated odds ratio after observing y. Then Bayes' Theorem yields

$$O^P_{n+1}(2) = \frac{p_H}{p_L} O^P_n, \quad O^Q_{n+1}(2) = \frac{q_H}{q_L} O^Q_n, \tag{6}$$

$$O^P_{n+1}(0) = \frac{1-p_H}{1-p_L} O^P_n, \quad O^Q_{n+1}(0) = \frac{1-q_H}{1-q_L} O^Q_n, \tag{7}$$

$$O^P_{n+1}(1) = \frac{(p_H(1-\overline{q}_n)+(1-p_H)\overline{q}_n)}{(p_L(1-\overline{q}_n)+(1-p_L)\overline{q}_n)} O^P_n, \quad O^Q_{n+1}(1) = \frac{(q_H(1-\overline{p}_n)+(1-q_H)\overline{p}_n)}{(q_L(1-\overline{p}_n)+(1-q_L)\overline{p}_n)} O^Q_n, \tag{8}$$

where we have written the posterior odds ratio as a product of the prior odds ratio and the likelihood ratio, which depends on y, and, importantly for $y = 1$, on P, Q. Denote the likelihood ratios by $L^P_y(Q)$ and $L^Q_y(P)$. The following result is immediate.

Lemma 2. *Posteriors behave as follows:*

$$\begin{aligned}
&\text{if } y = 2 \text{ then } P_{n+1} > P_n, \text{ and } Q_{n+1} > Q_n\\
&\text{if } y = 0 \text{ then } P_{n+1} < P_n, \text{ and } Q_{n+1} < Q_n\\
&\text{if } y = 1 \text{ then } P_{n+1} > P_n \text{ iff } \overline{q}_n < \frac{1}{2}, \text{ and } Q_{n+1} > Q_n \text{ iff } \overline{p}_n < \frac{1}{2}.
\end{aligned}$$

In other words, when the realization gives identified information about t and s, both agents update posteriors in the same way, as standard models suggest, though posteriors will still differ because priors differ. However, unidentified realizations ($y = 1$) can lead to divergent posteriors when beliefs are sufficiently polarized.

Discussion The reader may wonder about the simplicity of the model and signals received and whether our results are robust to more general setups. Our results are completely driven by the following two assumptions: (i) a partially identified model with (ii) relatively frequent observations that do not completely identify the model. We have deliberately chosen the simplest model to illustrate that partial identification can lead to beliefs diverging from the truth—in fact, if anything the simplicity of our model and the set of signals makes learning the truth more likely. There are many reasons to believe that the identification problem in a higher dimensional model with a richer signal-space would be more severe, making divergent learning even more likely, since fullyidentifying observations would be even less common. Furthermore, our agents can only possibly disagree about initial priors; polarization would be even more likely if agents also disagreed about the model parameters.

2.1 Theoretical Results

The main results of this section are as follows: when unidentified observations are relatively unlikely, beliefs converge to the truth (Proposition 1); when unidentified observations are relatively likely and the aggregate states are symmetric, then *with probability one* beliefs converge to confirm relative initial priors (Proposition 2); robustly, when unidentified observations are relatively likely there is a *positive probability* that beliefs converge to confirm relative initial priors (Proposition 5).

First, this process is path dependent: the order of observations matters for determining the posterior.

Lemma 3 (Path Dependence)**.** *The stochastic process defined by the posterior is* path dependent*. In other words, the posterior at any time* $T > 1$ *depends on the order of observations.*

The result is very intuitive. Since the evolution of posteriors depends on likelihood ratios, and since the likelihood ratios depend on priors, how an agent updates after seeing a one depends on what information has already been received. Naturally, path dependence is precisely a feature of the process. Agents interpret information differently—they have different likelihood functions—because their likelihood functions depend on their beliefs about the other variable. But these beliefs evolve as well. Thus, while our agents are completely rational and Bayesian (we have not made any behavioral assumptions) our model delivers learning that has a flavor of psychological short-sightedness: an agent need not recall the entire history of observations in order to update the posterior, but simply consults the current prior, which includes the information from the history of observations.

Our first result concerns a case where individuals converge asymptotically to the correct beliefs. Specifically, when $\theta = \sigma = 1$ or $\theta = \sigma = 0$, then beliefs converge to the truth. This is because identified observations ($y = 2$ or $y = 0$) are relatively most likely in these cases. Thus, for the most frequent observations, the likelihood ratios do not depend on both beliefs P and Q, and so agents tend to interpret observations the same most of the time.

Proposition 1. *If $\theta = \sigma = 1$ or $\theta = \sigma = 0$, then (respectively) $P_n, Q_n \overset{a.s.}{\to} 1$ and $P_n, Q_n \overset{a.s.}{\to} 0$.*

However, when intermediate observations ($y = 1$) are relatively likely, beliefs needs not converge to the truth. We now focus on these results.

2.1.1 Incorrect Beliefs in the Symmetric Case

While path dependence makes it more difficult to characterize the limiting properties of the process, we can still prove an important result when the states θ and σ are symmetric. Crucially, asymptotically beliefs converge to certainty confirming initial relative priors: extreme polarization occurs asymptotically.

Lemma 4. *Let $p_H = q_H$ and $p_L = q_L$. Then almost surely the ratio O^P/O^Q diverges to infinity if $P > Q$ and converges to zero if $P < Q$.*

The result follows because likelihood ratios get updated only after observing a one, going up or down depending only on the initial prior. The convergence of posteriors to reinforce initial priors immediately follows from this lemma.[12]

Proposition 2. *Let $p_H = q_H$ and $p_L = q_L$. Suppose the truth is $\theta = 0$ and $\sigma = 1$. If $P_0 > Q_0$, then $P_n \overset{a.s.}{\to} 1$ and $Q_n \overset{a.s.}{\to} 0$ (vice versa if $P_0 < Q_0$).*

Proof. By Lemma 4 the ratio of the odds-ratios, O^P/O^Q converges to infinity almost surely. This implies that $P_n \overset{\text{a.s.}}{\to} 1$ almost surely. Since $O^P/O^Q \to \infty$, at least one of $P_n \overset{\text{a.s.}}{\to} 1$ or $Q_n \overset{\text{a.s.}}{\to} 0$. However, if $Q = 0$ then the problem of Bayesian learning is isomorphic to learning whether $\sigma = 1$ given $P = 0$. That is, there exists $\varepsilon > 0$ such that if $P_n < \varepsilon$, then Q_n is a submartingale. By Doob's

[12] It is worth comparing this result with symmetry to Fryer et al. (2015). In their paper, there are two symmetric states (a and b) and agents' beliefs converge to one of these states to confirm their prior when ambiguous signals are sufficiently frequent. In this section, with symmetric parameters, the two polarized states are (1,0) and (0,1), and agents' beliefs will always converge to confirm their priors. In our model, polarization occurs because unidentified observations are frequent because the fundamental states are polarized. When the states are not polarized ((1,1) or (0,0)), then polarization will never happen.

Martingale Convergence Theorem, this problem converges to the truth ($Q = 1$) almost surely.[13] By symmetry, if $Q_n < \varepsilon$ then $P_n \overset{\text{a.s.}}{\to} 1$. □

Corollary 1 (Asymptotic belief divergence)**.** *Let $p_H = q_H$ and $p_L = q_L$. Suppose one agent holds prior beliefs with $P_0 > Q_0$ and the other has priors with $P_0 < Q_0$. Then if $\theta \neq \sigma$, with probability 1 agents' beliefs will asymptotically diverge to complete polarization, the first with $P_n \overset{a.s.}{\to} 1$ and $Q_n \overset{a.s.}{\to} 0$, and the reverse for the other.*

An implication of Proposition 1 and Lemma 4 is that when $\theta = 1 = \sigma$, both P and Q converge to 1. However, agents starting with different priors will continue to hold different *relative* beliefs about P and Q asymptotically—and those relative beliefs will diverge—even as beliefs converge to the truth (similar to Acemoglu et al., 2016). The symmetric result holds when $\theta = 0 = \sigma$.

The symmetric case is perhaps least interesting because the states $\theta = 1, \sigma = 0$ and $\sigma = 1, \theta = 0$ produce observationally equivalent outcomes. However, if agents could take actions whose payoffs depended on the values of θ and σ, disagreement would still matter. Nonetheless, this result is helpful to consider how divergence can occur even when states are asymmetric.[14]

2.1.2 Incorrect Beliefs in the Asymmetric Case

Individuals can converge asymptotically to false beliefs for any parameters. For any parameters there exist priors such that almost surely beliefs converge to the wrong values. Crucially, there is a robust set of parameters such that beliefs diverge. Throughout suppose $\theta = 0$ and $\sigma = 1$. (By symmetry all results also hold for $\theta = 1$ and $\sigma = 0$.)

First, we can characterize conditions on beliefs P, Q such that beliefs converge incorrectly almost surely. Ignoring a (measurably) small set of parameter values, we obtain a classification result for the asymptotic behavior of (P_n, Q_n). Proposition 3 states conditions on parameters that

[13] See Diaconis and Freedman (1986).

[14] Notice, for all parameter values, $L_1^P(Q)/L_1^Q(P)$ is increasing in P (this follows immediately from differentiation). Thus, there exists P/Q such that O^P/O^Q increases after a $y = 1$, and thus P/Q increases. When the $p's$ and $q's$ are close, then a low P/Q is required for $L_1^P(Q)/L_1^Q(P) > 1$ (for example, we saw in the symmetric case that this is always true). Farther from symmetry requires a higher P/Q. If beliefs fall in this region, if likelihoods for $y = 0, 2$ are not so different, if $y = 1$ are sufficiently likely—then beliefs are likely to diverge as in the symmetric case. Thus, asymptotic disagreement is less likely, and is only likely to occur if the initial priors are very different.

guarantee when priors exists such that beliefs will asymptotically converge to the endpoints (zero or one) with positive probability. Specifically, the proposition states that the asymptotic behavior of (P_n, Q_n) behavior can be stated in terms of the expectation of the transition function (i.e., the log odds ratio, defined in the Appendix) in neighborhoods of the extremal points of $[p_L, p_H] \times [q_L, q_H]$, because asymptotically individuals will accumulate in these neighborhoods with positive probability. (The conditions are technical, so the reader may want to skip the equations and instead consider Figure 1.)

Proposition 3. *The following hold:*

1. *If* $\mathbb{E}\left[\Delta \log O^P_{n+1}(q_L)\right] > 0$ *and* $\mathbb{E}\left[\Delta \log O^Q_{n+1}(p_H)\right] < 0$*, then there exist* (P_0, Q_0) *such that* $P_n \to 1$ *and* $Q_n \to 0$ *with positive probability tending to* 1 *as* $Q_0 \to 0$ *and* $P_0 \to 1$
2. *If* $\mathbb{E}\left[\Delta \log O^P_{n+1}(q_L)\right] < 0$*, then* $P \overset{a.s.}{\to} 0$*; if additionally* $\mathbb{E}\left[\Delta \log O^Q_{n+1}(p_L)\right] < 0$ *then* $Q \overset{a.s.}{\to} 0$*, whereas if* $\mathbb{E}\left[\Delta \log O^Q_{n+1}(p_L)\right] > 0$ *then* $Q \overset{a.s.}{\to} 1$.
3. *If* $\mathbb{E}\left[\Delta \log O^Q_{n+1}(p_H)\right] > 0$*, then* $Q \overset{a.s.}{\to} 1$*; if additionally* $\mathbb{E}\left[\Delta \log O^P_{n+1}(q_L)\right] < 0$ *then* $P \overset{a.s.}{\to} 0$*, whereas if* $\mathbb{E}\left[\Delta \log O^P_{n+1}(q_L)\right] > 0$ *then* $P \overset{a.s.}{\to} 1$.

Most importantly, if parameters satisfy condition 1 of Proposition 3 then there exist priors (correctly ordered) such that beliefs will diverge with positive probability. Numerically evaluating over all combinations $(p_H, p_L, q_H, q_L) \in (0,1)^4$ together with ordering restrictions, approximately 27.26% of parameters satisfy condition 1, meaning that for at least this many parameters there is a positive probability of converging to the wrong values for some priors.

Next, given parameters p_H, p_L, q_H, q_L, let $f(p,q) = \Pr\left((P_n, Q_n) \to (1,0) | P_0 = p, Q_0 = q\right)$ be the probability of that beliefs converge to $(1,0)$, which are the wrong beliefs. We have the following important results, including a finding on continuity in Proposition 4 that extends to a broader class of random dynamical systems with a Bernoulli Shift as a random component (see Proposition 1.6 in the Online Appendix).

Proposition 4. *The convergence probability* $f(p,q)$ *is continuous in priors* p, q.

Corollary 2. *If*

$$\frac{\log \frac{q_H}{q_L}}{\log \frac{p_H}{p_L}} < \frac{\log \frac{1-q_H}{1-q_L}}{\log \frac{1-p_H}{1-p_L}}, \tag{9}$$

then either $f(p,q) > 0$ *for all* $(p,q) \in (0,1) \times (0,1)$*, or it vanishes for all* (p,q).

Thus, if parameters satisfy condition 1 of Proposition 3 as well as equation (9), then for any (rightly ordered) priors, beliefs will diverge with positive probability. By simple counting of parameters and conditions, we have the following result.

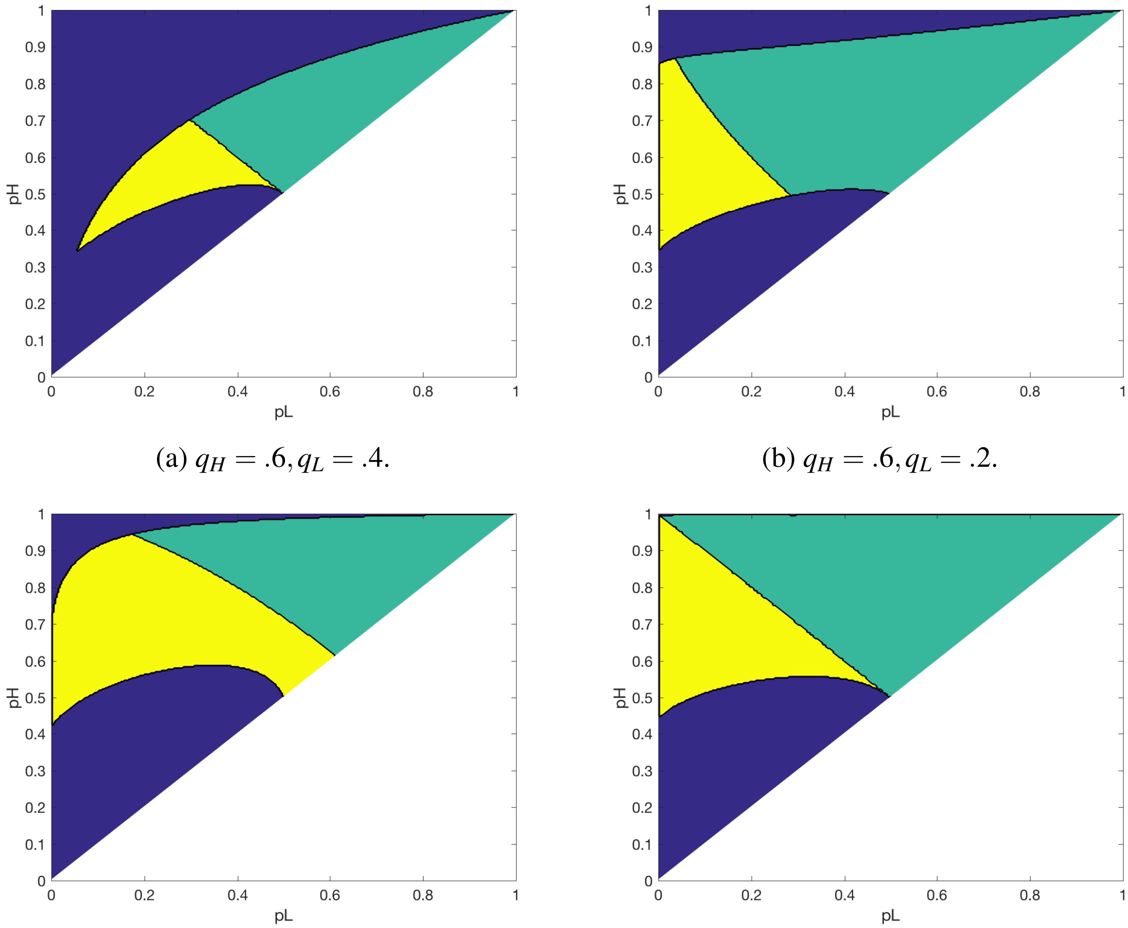
Proposition 5. *There is a robust set of parameters such that for any priors* $0 \leq Q_0 < P_0 \leq 1$ *beliefs diverge with positive probability.*

Approximately 8.33% of all parameters satisfy both sets of conditions. Figure 1 plots the sets of (p_H, p_L) parameters that satisfy the conditions, with each frame holding fixed a different (q_H, q_L). Yellow sections indicate that any priors will converge to (1,0) with positive probability. Green sections indicate that some priors are guaranteed to converge to (1,0) with positive probability. The blue sections are not guaranteed to never converge to (1,0), but the regions do not satisfy the conditions guaranteed that they might.

2.2 Simulation Results

Simulations evidence is useful for characterizing the limiting behavior of this process (which, again, is path dependent). In light of our theoretical results, we consider simulations with $\theta = 0$ and $\sigma = 1$ (the order does not matter), and consider two sets of parameter robustness.[15] We first set $q_H = .65, q_L = .4, p_L = .4$, and we do parameter sensitivity varying $p_H \in [.5, .9]$. We set prior beliefs to $P_0 = .8$ and consider $Q_0 = .4, .6, .75$. For each set of parameters we solve one million simulations, running simulations until beliefs converge to zero or one, and calculate the fraction

[15] Remember that as with standard Bayesian learning, beliefs converge to 1 or 0 with sufficient observations; however, the belief need not converge to the *true* value. As proved, when $\theta = \sigma = 1$ so that values of $y = 2$ are very likely (or when $\theta = \sigma = 0$ so that values of $y = 0$ are very likely), then beliefs converge to the truth, regardless of initial priors. However, convergence is rarely the case when $\theta \neq \sigma$.



(a) $q_H = .6, q_L = .4$. (b) $q_H = .6, q_L = .2$. (c) $q_H = .8, q_L = .4$. (d) $q_H = .8, q_L = .2$.

Figure 1: Parameters satisfying Proposition 3 and equation (9). Yellow: all priors will converge to (1,0) with positive probability. Green: some priors will converge to (1,0) with positive probability.

of simulations converging to one (the wrong value). Figure 2 plots the frequency of simulations that converge to the incorrect value. The simulation evidence suggests that for a broad range of parameters $P \to 1$ with high probability. The more symmetric are the states (the q's and p's are more similar), the more likely do beliefs converge to the wrong value. This occurs whenever $P_0 > Q_0$, regardless of the level. However, the more different are P_0 and Q_0, the greater probability of converging to $P = 1$ (the curve is a wider bell).

If the states are not so different and $P_0 > Q_0$ (the probability that $\theta = 1$ is greater than the probability that $\sigma = 1$), then the simulations suggest that asymptotically $P \to 1$ and $Q \to 0$ with

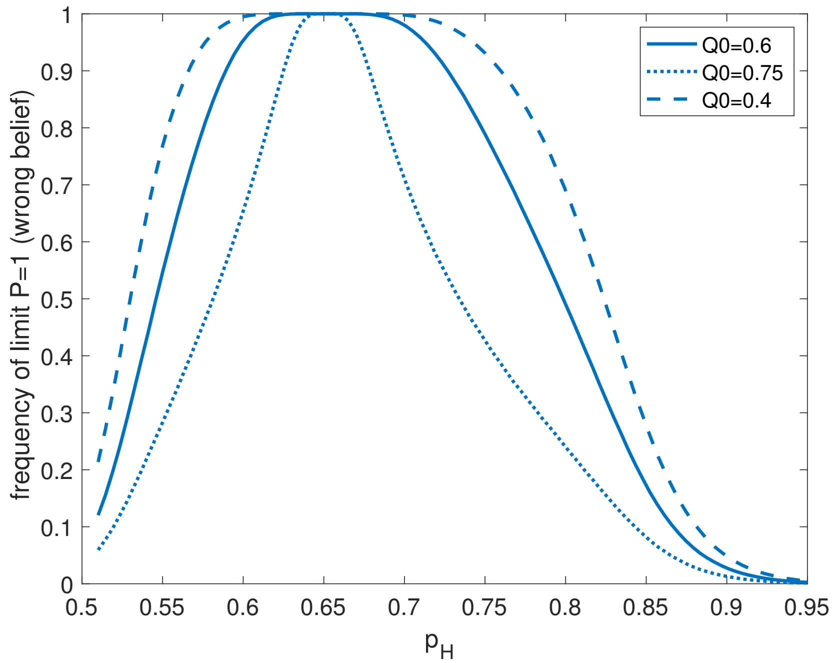


Figure 2: Probability of converging to wrong beliefs when starting with $P_0 > Q_0$.

high probability. This effect induces asymptotic disagreement when one agent has prior $P_0 > Q_0$ and the other has prior with $P_0 < Q_0$, and the disagreement is greater with more observations. When the parameters differ (p's and q's), the frequency of convergence decreases (continuously) as they differ by more. And thus the probability of permanent belief divergence decreases.[16]

When the states are far from symmetric, the qualitative result stands (there is a robust region with bell-shaped positive probability of converging to the wrong beliefs) but there need not be a set of parameters where asymptotic disagreement occurs with probability one. To further investigate this possibility, we run simulations varying p_H with $q_H = .4, q_L = .2$ but this time with $p_L = 1 - p_H$, again considering $Q_0 = .4, .6, .75$. Figure 3 plots these results for when the states are quite asymmetric. As one expects, the probability of converging to the wrong belief is greater when the

[16]Compare these results to Fryer et al. (2015). In their model, the probability of polarization is an increasing function of the probability of ambiguous signals. As our simulation results suggest, in our model the probability of polarization is a function of how different are parameters for each state. The more parameters differ, the easier it is to "statistically identify" observations of 1, which are otherwise unidentifiable. In their model ambiguous signals are completely unidentified. Thus, in our model, fundamentals determine the severity of the identification problem and the probability of polarization, whereas in their model the severity of ambiguity determines the probability of polarization.

initial belief Q_0 is lower.

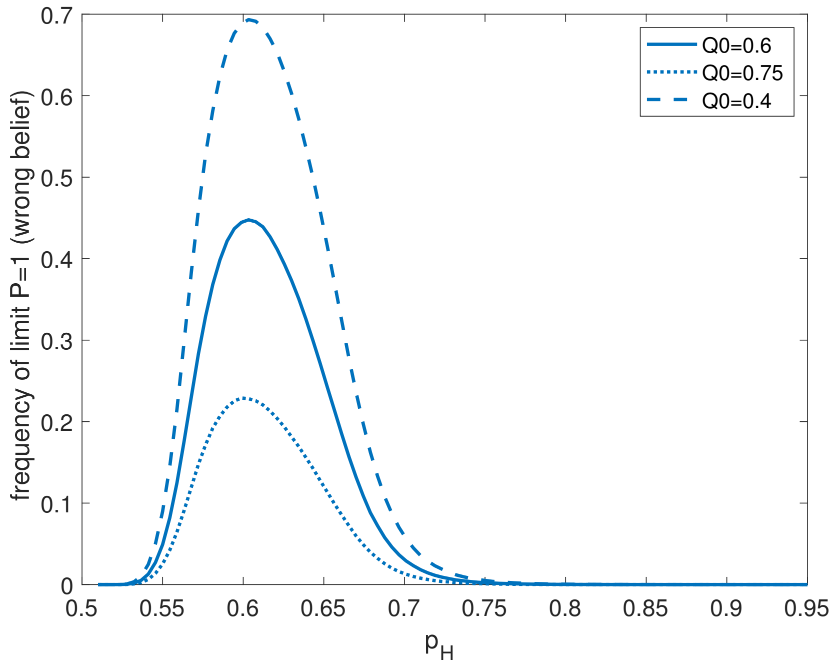


Figure 3: Probability of converging to wrong beliefs when starting with $P_0 > Q_0$. $q_H = .4, q_L = .2, p_L = 1 - p_H$.

Figure 4 plots the convergence probability over a grid of (p_H, p_L), holding fixed priors and parameters $q_H = .65, q_L = .4$. (Remember that the convergence probability is continuous in priors.) Not surprisingly given the previous figures, the convergence probability looks "multivariate Normal" as a function of the parameters.

While the values of the aggregate states are complements, posteriors need not converge to complements (i.e., they could both converge to 1 or to 0). Of course, this is not all that surprising since posteriors are not converging to the truth. The frequency of these convergences—of $P_n, Q_n \overset{\text{a.s.}}{\to} 1, 1$ and $P_n, Q_n \overset{\text{a.s.}}{\to} 0, 0$ are plotted in Figure 5 in the appendix.

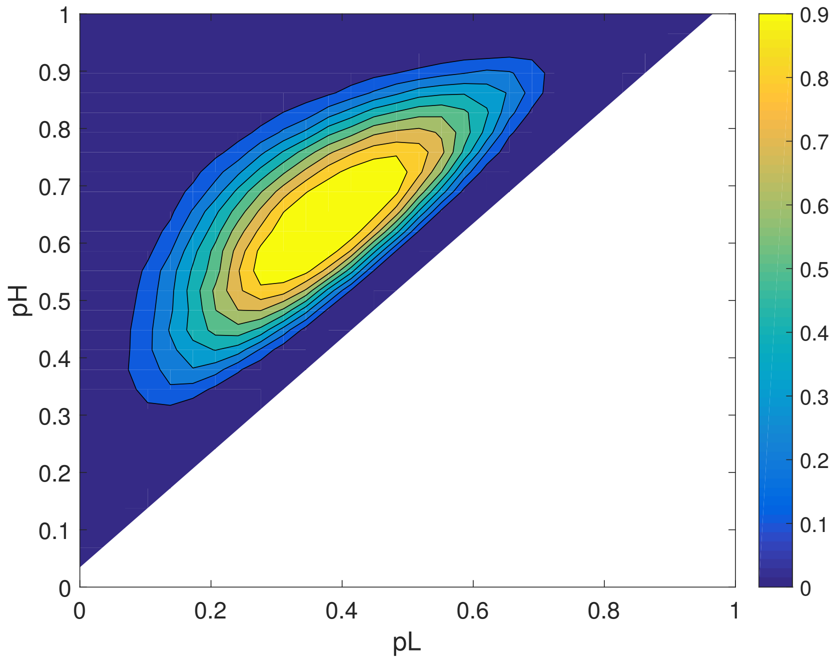


Figure 4: Probability of converging to wrong beliefs given prior $(P_0, Q_0) = (.8, .6)$, with $q_H = .65, q_L = .4$, varying p_H, p_L.

3 Implications

Our analysis has several implications for the effectiveness of attempts to relieve identification problems, polarization and information disclosure, and model specification and dynamics.

3.1 Actions

First, incomplete attempts to relieve the identification problem need not completely cure it. Suppose agents can choose with rare frequency to take an action leading to some realization: action a has a greater success rate when $\theta = 1$ and action b has a greater success rate when $\sigma = 1$. Since agents' beliefs are unlikely to converge, agents are likely to disagree about the best action to take. If agents could choose actions frequently enough, then their "experimentation" could provide richer data, perhaps even relieving the identification problem. However, this is precisely the problem that we find ourselves in the real world: we could know the effect of policies on macro variables if we could experiment frequently enough, but of course we cannot. Given unidentified models, it is no surprise that economists and politicians will continue to make contrasting recommendations. In some cases it is not surprising that recommendations will become increasingly polarized with more observations.

In fact, even frequent experimentation would still likely lead to polarization. First, if agents first observe many "unidentified observations," then beliefs will already be polarized and so identified observations will have small effects changing beliefs. Second, if the frequency of identified observations is exceeded by the frequency of unidentified observations (again, this is the scenario of concern), then polarization should still occur in the long run. An implication is that if economists produce a handful of well-designed natural experiments or instrumental variables to relieve an identification problem, the profession may nonetheless continue to hold divergent beliefs so long as there are plenty of cases in which the model is unidentified.[17]

3.2 Persuasion and Information

Our results have implications for persuasion and information release. First and most immediately, our model predicts that in some cases polarization may be unavoidable when initial beliefs are relatively different. Initial heterogeneity can lead to extreme polarization, and more information worsens polarization. This may explain why polarization in many places has seemingly increased in recent decades as information becomes more plentiful. If it is possible that priors can get "reset" (perhaps with a new generation), beliefs may be determined and unchanging until such a generational change occurs.

Second, our model is silent about where prior beliefs come from, but our results demonstrate that initial priors are critical for how people infer data (perhaps there are behavioral or generational explanations for priors; see Bénabou and Tirole (2016) for an overview of belief production). Our results suggest that the stakes from being able to control or influence initial priors (if that can be done) can be very high.

Third, our results suggest that agents attempting to persuade others (see Kamenica and Gentzkow,

[17] We acknowledge Bruce Sacerdote for this observation.

2011) may choose to purposely disclose "unidentified information" when priors are in their favor, knowing that people will infer the unidentified information in ways that are favorable to their outcome.[18] Additionally, in light of information undermining agents' initial beliefs, agents may have an incentive to "modify" an existing model by adding additional dimensions. This creates an identification problem, so that an initial hypothesis or theory can accommodate information that was initially at odds.

3.3 Model Specification and Dynamics

Finally, our results suggest that model specification has important implications for equilibrium dynamics, whether an economy has a representative agent or heterogeneous agents. When learning processes are unidentified, then a representative agent need not learn the truth, and beliefs among heterogeneous agents may become polarized rather than converging or being stable. Identification problems and learning can have important implications for macroeconomic dynamics (Collin-Dufresne et al., 2016; Milani, 2007), asset pricing (Adam et al., 2016), stability of non-rational expectations (Woodford, 2013) and strategies (Kalai and Lehrer, 1993), convergence of dynamical systems and dynamic models (Fernández-Villaverde et al., 2006; Schenk-Hoppé and Schmalfuß, 2001), and for properties of Bayesian estimation of macro models (Schorfheide, 2011) (the sensitivity of Bayesian estimation to initial priors may in some cases be much higher than commonly thought).

How can polarization be avoided? Our results show that, when agents face an identification problem and when the underlying fundamentals of the world are "polarized," agents' beliefs can diverge to certain but differing beliefs about fundamentals. However, a critical driver of this result is that agents update their beliefs whenever they get new information. If agents instead updated beliefs in the classical sense of Doob, "suspending judgment" until all the information has arrived, or completely revisiting their beliefs in light of all information, then with probability one polarization would not occur. (When agents form P_{n+1} using P_0 and $n+1$ observations, then P_{n+1} converges

[18] To the extent that priors differ because of initial private information, polarization can be prevented only when valuable communication can occur. A large literature since Crawford and Sobel (1982) have studied strategic information transmission.

to the truth for all but a measure zero of parameters—for example, the perfectly symmetric case.) In our model, agents update their beliefs frequently and rationally, and they rationally incorporate new information in light of their most recent prior. While this is a perfectly rational way of incorporating new information, it is also the reason that beliefs can become polarized. Further research should determine how agents form beliefs in light of new information and under what conditions those learning processes can be manipulated or modified.

4 Conclusion

The world is multi-dimensional—there are a number of factors that contribute to what we see—but the data we see is nearly always less dimensional than the world. We live with identification problems. As a result, Bayesian learning need not converge to the truth, and so agents with differing priors may have posteriors diverge forever, with greater divergence the more information received. When unidentified observations are sufficiently likely, beliefs are likely to converge to certainty confirming relative initial beliefs. Thus, asymptotic polarization can rationally occur.

We have characterized the limiting properties of beliefs for a simple model in which some observations are not identified. Our main result is that when unidentifiable observations are relatively more frequent, then asymptotically people's initial beliefs get reinforced: whatever state they initially believed was more likely to be good will with positive probability be believed to be the good state with certainty. Thus, beliefs become polarized in light of common information. However, if clearly identified observations are relatively likely, beliefs converge to the truth with probability one, and polarization will not occur.

References

ACEMOGLU, D., V. CHERNOZHUKOV, AND M. YILDIZ (2016): "Fragility of asymptotic agreement under Bayesian learning," *Theoretical Economics*, 11, 187–225.

ADAM, K., A. MARCET, AND J. P. NICOLINI (2016): "Stock market volatility and learning," *The Journal of Finance*, 71, 33–82.

ANDREONI, J. AND T. MYLOVANOV (2012): "Diverging opinions," *American Economic Journal: Microeconomics*, 209–232.

BALIGA, S., E. HANANY, AND P. KLIBANOFF (2013): "Polarization and ambiguity," *The American Economic Review*, 103, 3071–3083.

BATSON, C. D. (1975): "Rational processing or rationalization? The effect of disconfirming information on a stated religious belief." *Journal of Personality and Social Psychology*, 32, 176.

BÉNABOU, R. AND J. TIROLE (2016): "Mindful economics: The production, consumption, and value of beliefs," *The Journal of Economic Perspectives*, 30, 141–164.

BENOÎT, J.-P. AND J. DUBRA (2014): "A Theory of Rational Attitude Polarization," *Available at SSRN 2529494*.

CHAIKEN, S. ET AL. (1992): "Defensive processing of personally relevant health messages," *Personality and Social Psychology Bulletin*, 18, 669–679.

COLLIN-DUFRESNE, P., M. JOHANNES, AND L. A. LOCHSTOER (2016): "Parameter learning in general equilibrium: The asset pricing implications," *The American Economic Review*, 106, 664–698.

CRAWFORD, V. P. AND J. SOBEL (1982): "Strategic information transmission," *Econometrica: Journal of the Econometric Society*, 1431–1451.

DARLEY, J. M. AND P. H. GROSS (1983): "A hypothesis-confirming bias in labeling effects." *Journal of Personality and Social Psychology*, 44, 20.

DIACONIS, P. AND D. FREEDMAN (1986): "On the Consistency of Bayes Estimates," *Ann. Statist.*, 14, 1–26.

DIXIT, A. K. AND J. W. WEIBULL (2007): "Political polarization," *Proceedings of the National Academy of Sciences*, 104, 7351–7356.

EYSTER, E., A. GALEOTTI, N. KARTIK, AND M. RABIN (2014): "Congested observational learning," *Games and Economic Behavior*, 87, 519–538.

EYSTER, E. AND M. RABIN (2010): "Naive herding in rich-information settings," *American economic journal: microeconomics*, 2, 221–243.

FERNÁNDEZ-VILLAVERDE, J., J. F. RUBIO-RAMÍREZ, AND M. S. SANTOS (2006): "Convergence properties of the likelihood of computed dynamic models," *Econometrica*, 74, 93–119.

FRYER, R. G., P. HARMS, AND M. O. JACKSON (2015): "Updating Beliefs when Evidence is Open to Interpretation: Implications for Bias and Polarization," *Available at SSRN 2263504*.

HEIDHUES, P., B. KOSZEGI, AND P. STRACK (2015): "Unrealistic Expectations and Misguided Learning," *Available at SSRN*.

HIRSHLEIFER, D. AND S. H. TEOH (2003): "Limited attention, information disclosure, and financial reporting," *Journal of accounting and economics*, 36, 337–386.

JERN, A., K.-M. K. CHANG, AND C. KEMP (2014): "Belief polarization is not always irrational." *Psychological review*, 121, 206.

KALAI, E. AND E. LEHRER (1993): "Rational learning leads to Nash equilibrium," *Econometrica: Journal of the Econometric Society*, 1019–1045.

KAMENICA, E. AND M. GENTZKOW (2011): "Bayesian persuasion," *The American Economic Review*, 101, 2590–2615.

KONDOR, P. (2012): "The more we know about the fundamental, the less we agree on the price," *The Review of Economic Studies*, 79, 1175–1207.

LORD, C. G., L. ROSS, AND M. R. LEPPER (1979): "Biased assimilation and attitude polarization: The effects of prior theories on subsequently considered evidence." *Journal of personality and social psychology*, 37, 2098.

MILANI, F. (2007): "Expectations, learning and macroeconomic persistence," *Journal of monetary Economics*, 54, 2065–2082.

MUNRO, G. D. AND P. H. DITTO (1997): "Biased assimilation, attitude polarization, and affect in reactions to stereotype-relevant scientific information," *Personality and Social Psychology Bulletin*, 23, 636–653.

ORTOLEVA, P. AND E. SNOWBERG (2015): "Overconfidence in political behavior," *The American Economic Review*, 105, 504–535.

PEW RESEARCH CENTER (2014): "Stark Racial Divisions in Reactions to Ferguson Police Shooting," .

PLOUS, S. (1991): "Biases in the assimilation of technological breakdowns: Do accidents make us safer?" *Journal of Applied Social Psychology*, 21, 1058–1082.

RABIN, M. AND J. L. SCHRAG (1999): "First Impressions Matter: A Model of Confirmatory Bias," *The Quarterly Journal of Economics*, 114, 37–82.

SCHENK-HOPPÉ, K. R. AND B. SCHMALFUSS (2001): "Random fixed points in a stochastic Solow growth model," *Journal of Mathematical Economics*, 36, 19–30.

SCHORFHEIDE, F. (2011): "Estimation and evaluation of DSGE models: progress and challenges," Tech. rep., National Bureau of Economic Research.

SCHWARTZSTEIN, J. (2014): "Selective Attention and Learning," *Journal of the European Economic Association*, 12, 1423–1452.

SUNDARESAN, S. AND S. TURBAN (2014): "Inattentive valuation and belief polarization," Tech. rep., Working paper.

TABER, C. S. AND M. LODGE (2006): "Motivated skepticism in the evaluation of political beliefs," *American Journal of Political Science*, 50, 755–769.

USCHEL, J. (1995): "Poll: A Nation More Divided," USA Today, p. 5A.

WOODFORD, M. (2013): "Macroeconomic analysis without the rational expectations hypothesis," Tech. rep., National Bureau of Economic Research.

Appendices

A Proofs

Proof of Lemma 1. The posterior $\hat{P} > P$ if and only if $Q < \frac{1}{2}$, and the posterior $\hat{Q} > Q$ if and only if $P < \frac{1}{2}$. From the posteriors, $\hat{P} > P$ if $(1-Q) > P(1-Q) + (1-P)Q$, which is true if $1-Q > Q$. Similarly, $\hat{Q} > Q$ if $(1-P) > P(1-Q) + (1-P)Q$, which is true if $1-P > P$. □

Proof of Lemma 3. Consider the likelihood ratios for P (the results are symmetric for Q). Then the likelihood ratios are given by

$$L_2^P = \frac{p_H}{p_L}, \quad L_0^P = \frac{1-p_H}{1-p_L}, \quad L_1^P(Q) = \frac{p_H(1-\overline{q}) + (1-p_H)\overline{q}}{p_L(1-\overline{q}) + (1-p_L)\overline{q}}. \tag{10}$$

Thus, the likelihood from observing $y=0$ or $y=2$ is independent of Q, but the likelihood from observing $y=1$ depends on Q. Thus, the odds ratios after observing $(0,1)$ and $(1,0)$ are given by

$$O_2^P(0,1) = L_0^P L_1^P(Q_1) O_0^P, \quad \text{and} \quad O_2^P(1,0) = L_0^P L_1^P(Q_0) O_0^P, \tag{11}$$

where in the first case the likelihood ratio for the $y=1$ observation depends on the updated belief about Q after the $y=0$ observation, while in the second case the likelihood ratio for the $y=1$ observation depends on the initial prior belief about Q, which has not been updated. □

Proof of Lemma 4. First, since $p_H = q_H$ and $p_L = q_L$, the likelihood ratios given observations $y = 0$ and $y = 2$ are equal:

$$L_2^P = \frac{p_H}{p_L} = L_2^Q, \qquad L_0^P = \frac{1-p_H}{1-p_L} = L_0^Q.$$

Thus, O^P/O^Q is unchanged after these observations.

Second, O^P/O^Q increases after $y = 1$ whenever $P > Q$. Differentiating $L_1^P(Q)$ with respect to Q shows that it is decreasing in Q. By symmetry, if $P > Q$, then $L_1^P(Q)/L_1^Q(P) > 1$. Similarly, differentiating and collecting terms, $L_1^P(Q)/L_1^Q(P)$ is increasing in P. Thus, as P/Q grows, the ratio of likelihoods grows and thus O^P/O^Q increases by more. Thus, $L_1^P(Q_n)/L_1^Q(P_n) \geq L_1^P(Q_0)/L_1^Q(P_0) > 1$, and so $L_1^P(Q_n)/L_1^Q(P_n)$ is bounded below by a number strictly greater than 1. Since all probabilities are strictly positive, by the Strong Law of Large Numbers asymptotically there will be an infinite number of $y = 1$ observations, and thus O^P/O^Q increases without bound. By similar argument, if $P < Q$ then O^P/O^Q decreases to zero. □

Proof of Proposition 1. We prove for P_n (the proof for Q_n is similar). If $\theta = \sigma = 1$, then the inverse-odds ration $(O_n^P)^{-1} = \frac{1-P_n}{P_n}$ is upper bounded on Ω by the random variable $\prod_{i=1}^n r_i$ defined such that

$$r_i(\omega) = \begin{cases} \frac{p_L}{p_H} & \text{if } y_i(\omega) = 2 \\ \frac{1-p_L}{1-p_H} & \text{if } y_i(\omega) = 0 \\ \frac{p_L(1-q_H)+(1-p_L)q_H}{p_H(1-q_H)+(1-p_H)q_H} & \text{if } y_i(\omega) = 1 \end{cases}$$

Note that in this case

$$\mathbb{E}\left[\prod_{i=1}^{n+1} r_i \,\middle|\, r_n, r_{n-1}, \ldots\right] = \mathbb{E}\left[r_{n+1} | r_n, r_{n-1}, \ldots\right] \prod_{i=1}^{n} r_i = \prod_{i=1}^{n} r_i. \tag{12}$$

But by the Hewitt-Savage 0-1 Law, $\liminf_{n\to\infty} \prod_{i=1}^n r_i$ is a constant $c \in [0, \infty)$, and it is clear that such a constant is invariant under multiplication by $\frac{p_L}{p_H}, \frac{1-p_L}{1-p_H}$. Hence $c \in \{0, \infty\}$, but as (12) and the law of iterated expectations imply $\mathbb{E}\left[\prod_{i=1}^n r_i\right]$ is bounded, it must be true that $\liminf_{n\to\infty} \prod_{i=1}^n r_i = 0$. Hence, $\liminf_{n\to\infty} (O_n^P)^{-1} = 0$, and thus $\limsup_{n\to\infty} P_n = 1$ (all equalities hold a.e.). Then (16) implies that $P_n \overset{\text{a.s.}}{\to} 1$. □

Suppose for now that $\theta = \sigma = 1$ and $p_H, q_H > 0$. Consider the following recursion relations for (P_n) as a function of Q_n:

$$
\begin{aligned}
P_{n+1}(2) &= \frac{P_n p_H}{P_n p_H + (1-P_n)p_L} \\
P_{n+1}(0) &= \frac{P_n(1-p_H)}{P_n(1-p_H)+(1-p)(1-p_L)} \\
P_{n+1}(1) &= \frac{P_n\left(p_H(1-\overline{q}_n)+(1-p_H)\overline{q}_n\right)}{P_n\left(p_H(1-\overline{q}_n)+(1-p_H)\overline{q}_n\right)+(1-P_n)\left(p_L(1-\overline{q}_n)+(1-p_L)\overline{q}_n\right)},
\end{aligned}
$$

where $\overline{q}_n = Q_n q_H + (1-Q_n)q_L \in [q_L, q_H]$. One can determine that:

$$
\begin{aligned}
&\overline{q}_n\left(P_n p_H + (1-P_n)p_L\right)p_{n+1}(2) \\
&\quad + (1-\overline{q}_n)\left(P_n(1-p_H)+(1-p)(1-p_L)\right)p_{n+1}(0) \\
&\quad + \Big(P_n\left(p_H(1-\overline{q}_n)+(1-p_H)\overline{q}_n\right)+(1-P_n)\left(p_L(1-\overline{q}_n)+(1-p_L)\overline{q}_n\right)\Big)p_{n+1}(1) \\
&\quad = P_n, \qquad (13)
\end{aligned}
$$

i.e. the agent believes that his estimate of P_n will be correct in expectation in the next period. In addition, the inequality

$$
P_{n+1}(2) \geq P_{n+1}(1) \geq P_{n+1}(0) \qquad (14)
$$

naturally holds because $\frac{p_H}{1-p_H} > \frac{p_L}{1-p_L}$. Note that in comparison to (13), one has

$$
\begin{aligned}
&q_H p_H \geq \overline{q}_n\left(P_n p_H + (1-P_n)p_L\right) \\
&(1-q_H)(1-p_H) \leq (1-\overline{q}_n)\left(P_n(1-p_H)+(1-p)(1-p_L)\right).
\end{aligned}
$$

Hence, the imposition of (14) implies

$$\begin{aligned}\mathbb{E}_n\left(P_{n+1}\right) =& q_H p_H P_{n+1}(2) + (1-q_H)(1-p_H)P_{n+1}(0) \\ & + \left((1-q_H)p_H + q_H(1-p_H)\right) P_{n+1}(1) \\ \geq & P_n, \end{aligned} \tag{15}$$

where $\mathbb{E}_n(\cdot)$ is the time n expectation given $P_n, p_{n-1}, \ldots$. So (P_n) is a supermartingale, and by Doob's Martingale Convergence Theorem it is true that

$$(P_n) \overset{\text{a.s.}}{\to} \limsup_{n\in\mathbb{N}} P_n : [0,1] \to [0,1]. \tag{16}$$

In the following lemma, let $\mathscr{F}_n$ be the filtration of $\mathscr{F}_\infty$ which contain (as cylinder sets of $\mathscr{F}_\infty$) sets in the probability space Ω on which (P_n, Q_n) are defined. For $\omega \in \Omega$ write $\omega = \omega_1\omega_2\ldots$, where $\omega_n \in \{0,1,2\}$ according to the value of $y_n(\omega)$.

Lemma 5. *If* $\lim_{n\to\infty} P_n$ *exists in* $[0,1]$ *almost everywhere and* $p_H, p_L \in (0,1)$*, then*

$$Pr\left(\omega : \lim_{n\to\infty} P_n(\omega) \in (0,1)\right) = 0. \tag{17}$$

Proof. We claim that any point ω such that $\lim_{n\to\infty} P_n(\omega) \in (0,1)$ has $\omega_n \in \{0,2\}$ for only finitely many indices n. This is easy to see for $\omega_n = 2$: assume to the contrary that infinitely many $(n_m) \subset \mathbb{N}$ satisfy $\omega_{n_m} = 2$, and let $\lim_{n\to\infty} p_n(\omega) = c \in (0,1)$. Then taking such m arbitrarily high, $(P_{n_m+1}) \to \frac{cp_H}{cp_H+(1-c)p_L} > c$, a contradiction. The proof when $\omega_n = 0$ infinitely often is similar. (17) follows immediately by the Second Borel-Cantelli Lemma. ☐

Proof of Proposition 3

Consider the log-odds ratios, for which we have the following recursions:

$$\log O^P_{n+1}(2) = \log \frac{p_H}{p_L} + \log O^P_n \tag{18}$$

$$\log O^P_{n+1}(0) = \log \frac{1-p_H}{1-p_L} + \log O^P_n \tag{19}$$

$$\log O^P_{n+1}(1) = \log \frac{p_H(1-\overline{q}_n)+(1-p_H)\overline{q}_n}{p_L(1-\overline{q}_n)+(1-p_L)\overline{q}_n} + \log O^P_n. \tag{20}$$

Consider the random variable $\Delta \log O^P_{n+1}(\overline{q})$ (and the analogous expression for Q), whose expected value can be evaluated explicitly as:

$$\begin{aligned}\mathbb{E}\left[\Delta \log O^P_{n+1}(\overline{q})\right] =& p_L q_H \log \frac{p_H}{p_L} + (1-p_L)(1-q_H)\log\frac{1-p_H}{1-p_L} \\ &+ (p_L(1-q_H)+(1-p_L)q_H)\log\frac{p_H(1-\overline{q})+(1-p_H)\overline{q}}{p_L(1-\overline{q})+(1-p_L)\overline{q}}.\end{aligned}$$

The restriction of the transition function to the interior of the set $[p_L,p_H]\times[q_L,q_H]$ often is less relevant than its restriction to these endpoints.

We proceed in three lemmas, which deal with different cases in the parameters.

Lemma 6. *Let $\theta = 0$ and $\sigma = 1$. If $\mathbb{E}\left[\Delta \log O^P_{n+1}(q_L)\right] > 0$ and $\mathbb{E}\left[\Delta \log O^Q_{n+1}(p_H)\right] < 0$, then there exist (P_0,Q_0) such that $P_0 \to 1$ and $Q_0 \to 0$ with positive probability tending to 1 as $Q_0 \to 0$ and $P_0 \to 1$.*

Proof. First, recall Hoeffding's inequality: let $X_1,\ldots,X_n$ be independent and $a_i \le X_i \le b_i$ for all i. Then

$$\Pr\left(\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \ge n\varepsilon\right) \le \exp\left(\frac{-2n^2\varepsilon^2}{\sum_{i=1}^n (b_i-a_i)}\right). \tag{21}$$

By continuity, there is a $\delta_1 > 0$ such that if $\overline{q} < q_L + \delta_1$, then $\mathbb{E}\left[\Delta \log O^P_{n+1}(\overline{q})\right] > \varepsilon_1 > 0$. Consider the process (X_n) which has $\Delta(X_n) = \Delta(\log O^P_n)$ when $\omega_n = 0,2$, and $\Delta(X_n) = \log \frac{p_H(1-q_L-\delta_1)+(1-p_H)(q_L+\delta_1)}{p_L(1-q_L-\delta_1)+(1-p_L)(q_L+\delta_1)}$ when $\omega_n = 1$. Thus, $\mathbb{E}[\Delta(X_n)] > \varepsilon_1$. Also, if $\overline{q}_n \in [q_L, q_L +$

$\delta_1]$ for all values of n, we have the relation $\log O_n^P \geq X_n$. Set Q^* to satisfy $Q^* q_H + (1-Q^*) q_L = q_L + \delta_1$. By the analogous assumption for Q_n, there exists some $\delta_2 > 0$ and $\varepsilon_2 > 0$ such that $\mathbb{E}\left[\Delta \log O_{n+1}^Q(\overline{p})\right] < -\varepsilon_2$ if $\overline{p} > p_H - \delta_2$. Construct the random walk (Y_n) as an upper bound for $\log O_n^Q$ just as X_n was constructed as a lower bound for $\log O_n^P$, with $\overline{P}$ replaced with $p_H - \delta_2$. Set P^* to satisfy $P^* p_H + (1-P^*) p_L = p_H - \delta_2$. Also, let $X^* = \log \frac{P^*}{1-P^*}$ and $Y^* = \log \frac{Q^*}{1-Q^*}$. Note that if $X_n \geq X^*$ and $Y_n \leq Y^*$ for all n, then $P_n \geq P^*$ and $Q_n \leq Q^*$ for all n and consequentially X_n is always a lower bound for P_n whereas Y_n is always an upper bound for Q_n. Fix $X_0 = \log O_0^P$ and $Y_0 = \log O_0^Q$. By (21), for $\nu > 0$ we have

$$\begin{aligned}\Pr\left(X_n - X_0 \leq n\mathbb{E}\left[X_i\right] - n\nu\right) &= \Pr\left(\sum_{i=1}^{n}(\Delta X_i - \mathbb{E}\left[\Delta X_i\right]) \leq -n\nu\right) \\ &\leq \exp\left(\frac{-2n^2\nu^2}{\sum_{i=1}^{n}\left(\log\frac{p_H}{p_L} - \log\frac{1-p_H}{1-p_L}\right)}\right).\end{aligned}$$

Pick $X_0 > X^*$. Of course, $\Delta X_i > \varepsilon_1$, so picking $\nu = \varepsilon_1/2$, we have

$$\Pr\left(X_n \leq X^* + n\frac{\varepsilon_1}{2}\right) \leq \Pr\left(X_n \leq X_0 + n\frac{\varepsilon_1}{2}\right) \leq \exp\left(\frac{-\varepsilon_1^2 n\nu^2}{2\left(\log\frac{p_H}{p_L} - \log\frac{1-p_H}{1-p_L}\right)}\right),$$

and if $n \leq \left\lfloor \frac{X_0 - X^*}{-\log\frac{1-p_H}{1-p_L} + \varepsilon_1/2} \right\rfloor$, $\Pr\left(X_n \leq X^*\right) = 0$. Hence,

$$\Pr\left(X_n \leq X^* + \frac{n\varepsilon_1}{2} \text{ for some } n\right) \leq \sum_{\left\lfloor \frac{X_0 - X^*}{-\log\frac{1-p_H}{1-p_L} + \varepsilon_1/2} \right\rfloor}^{\infty} \exp\left(\frac{-\varepsilon_1^2 n\nu^2}{2\left(\log\frac{p_H}{p_L} - \log\frac{1-p_H}{1-p_L}\right)}\right).$$

Obviously, $X_0 = \log O_0^P$ can be chosen high enough so that this quantity is strictly less than $\frac{1}{2}$.

Similarly, we can choose $Y_0 = \log O_0^Q$ so low such that $\Pr\left(Y_n \geq Y^* - \frac{n\varepsilon_2}{2} \text{ for some } n\right) < \frac{1}{2}$. Then

$$\begin{aligned}
\Pr(P_n \to 1 \text{ and } Q_n \to 0) &\geq \Pr\left(X_n \geq X^* + \frac{n\varepsilon_1}{2} \text{ and } Y_n \leq Y^* - \frac{n\varepsilon_2}{n} \text{ for all } n\right) \\
&\geq 1 - \Pr\left(X_n \leq X^* + \frac{n\varepsilon_1}{2} \text{ for some } n\right) - \Pr\left(Y_n \geq Y^* - \frac{n\varepsilon_2}{2} \text{ for some } n\right) \\
&> 0.
\end{aligned}$$

Indeed, it is clear that as $Q_0 \to 0$ and $P_0 \to 1$ for a fixed P^* and Q^*, this probability tends to 1, despite the fact that $\theta = 0$ and $\sigma = 1$.

□

Lemma 7. *If* $\mathbb{E}\left[\Delta \log O_{n+1}^P(q_L)\right] < 0$, *then* $P \overset{a.s.}{\to} 0$, *and if* $\mathbb{E}\left[\Delta \log O_{n+1}^P(p_H)\right] > 0$, *then* $Q \overset{a.s.}{\to} 1$

Proof. The proof follows the method of Lemma 6 by setting $Q^* = 1$ and $P^* = 0$. □

Lemma 8. *Let* $\mathbb{E}\left[\Delta \log O_{n+1}^P(q_L)\right] < 0$. *Then* $P \overset{a.s.}{\to} 0$, *and if* $\mathbb{E}\left[\Delta \log O_{n+1}^Q(p_L)\right] < 0$ *then* $Q \overset{a.s.}{\to} 0$, *whereas if* $\mathbb{E}\left[\Delta \log O_{n+1}^Q(p_L)\right] > 0$ *then* $Q \overset{a.s.}{\to} 1$.

Alternately, if $\mathbb{E}\left[\Delta \log O_{n+1}^Q(p_H)\right] > 0$, *then* $Q \overset{a.s.}{\to} 1$ *and if* $\mathbb{E}\left[\Delta \log O_{n+1}^P(q_L)\right] < 0$ *then* $P \overset{a.s.}{\to} 0$, *whereas if* $\mathbb{E}\left[\Delta \log O_{n+1}^P(q_L)\right] > 0$ *then* $P \overset{a.s.}{\to} 1$.

Proof. The convergence $P \overset{\text{a.s.}}{\to} 0$ follows from Lemma 7. As for convergence of Q, if $\mathbb{E}\left[\Delta \log O_{n+1}^P(p_L)\right] < 0$, then $\mathbb{E}\left[\Delta \log O_{n+1}^P(\overline{p})\right]$ is upper-bounded by a strictly negative number everywhere, and a law of large numbers argument suffices. On the other hand, if $\mathbb{E}\left[\Delta \log O_{n+1}^P(p_L)\right] > 0$, then as P concentrates almost surely at 0, $\mathbb{E}\left[\Delta \log O_{n+1}^Q\right]$ becomes lower bounded (for all sufficiently high n) by a positive $\varepsilon \in (0, \mathbb{E}\left[\Delta \log O_{n+1}^P(p_L)\right])$ almost surely, whence $Q \overset{\text{a.s.}}{\to} 1$. An obvious symmetry establishes the second claim. □

Proof of Proposition 4

Here we provide the essential results necessary for the main claims of our paper. In the Internet Appendix we provide a proof of more general features of continuity, which builds on these results. In the proof, we discuss the relevance of the random dynamical theory of cocycles to our model, as in (Schenk-Hoppé and Schmalfuß, 2001). Our result of continuity also holds on a larger set of random dynamical systems with a Bernoulli shift as a random component.

Consider the function $f(p,q)$ giving the probability of convergence of (P_n, Q_n) to $(1,0)$ if $P_0 = p$, $Q_0 = q$ given p_H, p_L, q_H, q_L. This function has some nice properties. Suppose that $\theta = 0, \sigma = 1$. First,

$$\begin{aligned} f(p,q) =& p_L q_H f\left(P'((p,q),2), Q'((p,q),2)\right) + (1-p_L)(1-q_H)\left(P'((p,q),0), Q'((p,q),0)\right) \\ &+ \Big(p_L(1-q_H) + (1-p_L)q_H\Big) f\left(P'((p,q),1), Q'((p,q),1)\right). \end{aligned}$$

Furthermore, observe that for a *fixed* q, $f(p,q)$ is monotonically increasing in p:

Lemma 9. *For $p' \geq p$ and $q' \leq q$, $f(p',q') \geq f(p,q)$. Therefore, $f(\cdot,q)$ is continuous almost everywhere at any fixed q.*

Proof. Fix a history $\omega \in \Omega$. Let $P_n(\omega), Q_n(\omega)$ correspond to initial condition (p,q) and $P'_n(\omega), Q'_n(\omega)$ be defined to correspond to (p',q'). The claim can be verified by on the hypothesis that in each n, we have

$$P'_n(\omega) \geq P_n(\omega), \quad Q'_n(\omega) \leq Q_n(\omega). \tag{22}$$

The argument is accomplished with the observation that

1. If $y_n(\omega) = 0, 2$, the ordering is preserved by monotonicity of the relevant functions.

2. If $y_n(\omega) = 1$, then a smaller value of Q_n corresponds with a larger increase in P_n. Conversely, a larger value of P_n corresponds with a smaller increase (larger decrease) in Q_n. This can be easily verified by noting that the odds ratio O^P_{n+1} increases by more when $y_n = 1$ and Q_n is smaller. Conversely, O^Q_{n+1} increases by less when $y_n = 1$ and P_n is larger. Because $O^P_{n+1} \geq O^{P'}_{n+1}$ if and only if $P_{n+1} \geq P'_{n+1}$, the inequalities in (22) are indeed preserved.

□

Proof of Corollary 2. Note that

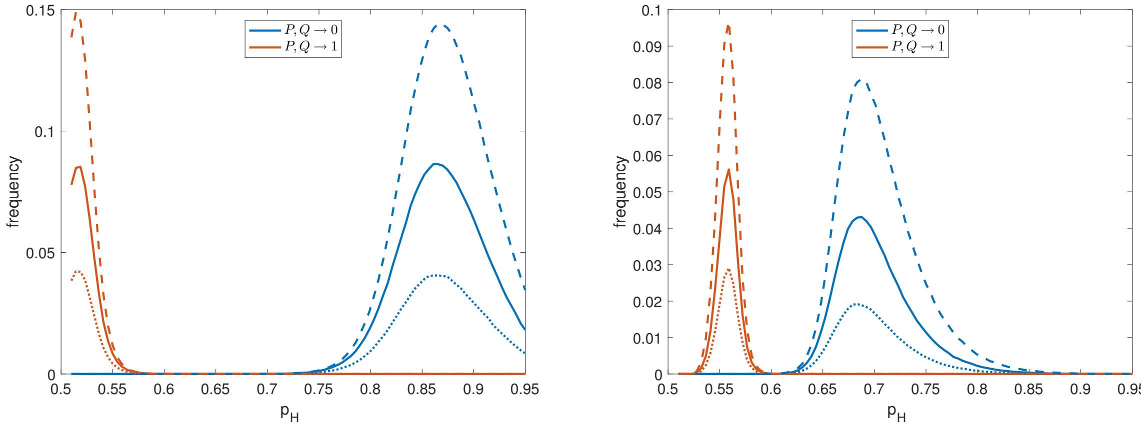
$$n\log\frac{p_H}{p_L}+\left[-n\frac{\log\frac{q_H}{q_L}}{\log\frac{1-q_H}{1-q_L}}+1\right]\log\frac{1-p_H}{1-p_L}=n\left(\log\frac{p_H}{p_L}+\left(-\frac{\log\frac{q_H}{q_L}}{\log\frac{1-q_H}{1-q_L}}+o(1)\right)\log\frac{1-p_H}{1-p_L}\right)$$

$$n\log\frac{q_H}{q_L}+\left[-n\frac{\log\frac{q_H}{q_L}}{\log\frac{1-q_H}{1-q_L}}+1\right]\log\frac{1-q_H}{1-q_L}<0.$$

Hence, there must exist positive integers n, $m\approx -n\frac{\log\frac{q_H}{q_L}}{\log\frac{1-q_H}{1-q_L}}$ such that $n\log\frac{p_H}{p_L}+m\log\frac{1-p_H}{1-p_L}>0$, and $n\log\frac{q_H}{q_L}+m\log\frac{1-q_H}{1-q_L}>0$. It follows that if $y=2$ for $n\ell$ times and $y=0$ for $m\ell$ times, as ℓ becomes arbitrarily large, $\log O^P_{\ell(n+m)}$ becomes arbitrarily large and $\log O^Q_{\ell(n+m)}$ becomes arbitrarily small from any initial prior (P,Q). In particular, if f does not vanish for all (p,q), by Lemma 9, there is a critical O^{P*} and O^{Q*} such that for all pairs (O^P,O^Q) with $O^P\geq O^{P*}$ and $O^Q\leq O^{Q*}$, there is a positive probability that $O^P\to\infty$. Because this critical threshold can be reached in a finite number of steps from any prior, any prior has a positive probability of $O^P\to\infty$. The second claim is proved similarly by exchanging 'P' and 'Q'. □

B Additional Figures

When states are not symmetric, posteriors can converge to values that are not even "symmetric" with the truth. Figure 5 shows the likelihood that both beliefs converge to zero in this case. The curve shifts up (more likely) when P_0 is more different from Q_0, down the closer they are, but is not much affected by the levels. This result is not surprising: the Q-belief will get pushed to zero when there are a sequence of $y=0$ which updates Q down, which is likely to occur if P is high.



(a) $q_H = .6, q_L = p_L = .4$.

(b) $q_H = .4, q_L = .2, p_L = 1 - p_H$.

Figure 5: Probability both $P \to 0, Q \to 0$ when starting with $P_0 > Q_0$.