March 3, 2010

Class # 10: The Poisson Distribution

Reading: Section 3.6 of Devore

The Poisson probability distribution is used to model the count of very rare events. If $X$ has a Poisson distribution with rate parameter $\lambda$ (this is denoted as $X \sim \text{Poi}(\lambda)$), then $X$ has probability mass function $p(x)$ where

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}.$$ 

Note that $E(X) = \lambda$ and $\text{Var}(X) = \lambda$.

Homework: These problems are not due (because of the exam), but they are recommended.

2. Devore, page 114, #50.
3. Devore, page 114, #56(a), (b), (c) and (e).
4. Devore, page 115, #60.

5. A multiple-choice test consists of 20 items, each with four choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices. A passing grade is 12 items or more correct.

   (a) What is the probability that the student passes?
   (b) Answer the question in part (a) again, assuming that the student can eliminate two of the choices on each question.

Homework Solutions:

1. (a) $E(Y) = \sum_{y=0}^{3} yp(y) = 0 \times .6 + 1 \times .25 + 2 \times .10 + 3 \times .05 = .6$
   (b) $E[100Y^2] = 100E[Y^2]$, where $E(Y^2) = \sum_{y=0}^{3} y^2p(y) = 1.1$. This makes the answer 110.

2. (a) $E(X) = \sum_{x} xp(x) = 16.3$, $E(X^2) = \sum_{x} x^2p(x) = 272.98$, and $V(X) = E(X^2) - [E(X)]^2 = 3.9936$
   (b) $E[25X - 8.5] = 25 \times E(X) - 8.5 = 401$
   (c) $\text{Var}(25X - 8.5) = 25^2 \text{Var}(X) = 2496$
   (d) $E[h(X)] = E(X) - .01E(X^2) = 16.38 - .01 \times 272.298 = 13.65$

3. This question is kind of tricky because you have to keep in mind that if the market only purchases 3 magazines, for example, then it only has three to sell. You thus have to adjust the probability mass function.

   Say the market only purchases 3 magazines. In this case, $p(1) = 1/15, p(2) = 2/15$, and $p(3)$ (which is really the probability that at least three will be demanded) is 12/15. In this case the expected revenue can be calculated as $2 \times E(X) - 3$ where $E(X) = 1 \times 1/15 + 2 \times 2/15 + 3 \times 3/15 + 4 \times 9/15$. The expected revenue is 2.46667.

   Say the market purchases 4 magazines. In this case, $p(1) = 1/15, p(2) = 2/15, p(3) = 3/15$ and $p(4) = 9/15$. In this case, $E(X) = 1 \times 1/15 + 2 \times 2/15 + 3 \times 3/15 + 4 \times 9/15$, and the expected revenue can be calculated as $2 \times E(X) - 4 = 2.66667$. The purchase of four magazines is thus preferred.
4. \[ E(X) = \sum_{x=1}^{n} xp(x) = \frac{1}{n} \sum_{x=1}^{n} x = \frac{n(n+1)}{2n} = \frac{n+1}{2} \] and \( Var(X) = E(X^2) - E(X)^2 \) where \( E(X^2) = \sum_{x=1}^{n} x^2 p(x) = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{(n+1)(2n+1)}{6} \). \( Var(X) \) is thus \( Var(X) = \frac{(n+1)(2n+1)}{6} - \frac{n^2}{4} \).

5. Let \( Y = -X \). The pmf of \( Y \) is

<table>
<thead>
<tr>
<th>( y )</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>-4</th>
<th>-5</th>
<th>-6</th>
</tr>
</thead>
</table>

It is easy to show that \( Var(Y) = Var(X) \)