Class # 14: Joint Distributions

**Reading:** Section 5.1 of Devore

Two discrete random variables, $X$ and $Y$, follow the joint probability mass function $p(x, y)$ (written $(X, Y) \sim p(x, y)$) if

$$P(X = x, Y = y) = p(x, y) \quad \forall \ x, y.$$  

The marginal distributions of both $X$ and $Y$ can easily be computed from the joint probability mass function using the formulas given below:

- $p(x) = P(X = x) = \sum_{y} p(x, y)$
- $p(y) = P(Y = y) = \sum_{x} p(x, y)$

Two continuous random variables, $X$ and $Y$, follow the joint probability distribution function $f(x, y)$ (written $(X, Y) \sim f(x, y)$) if

$$P((X, Y) \in A) = \int_{(x,y)\in A} f(x,y) dx dy.$$  

The marginal distributions of both $X$ and $Y$ can easily be computed from the joint probability mass function using the formulas given below:

- $f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$
- $f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$

**Homework:** These problems are due at the beginning of class on 3/12/10.

1. Devore, page 154, #32.
2. Devore, page 155, #38
3. Suppose that in a certain population, individuals’ heights are approximately normally distributed with parameters $\mu = 70$ in. and $\sigma = 3$ in.
   
   (a) What proportion of the population is over 6 feet tall?
   (b) 35% of the population is over how many feet tall?
4. Suppose that the measured voltage in a certain electric circuit has a normal distribution with mean 120 and standard deviation 2. If three independent measurements of the voltage are made, what is the probability that all three measurements will lie between 116 and 118?
5. Evaluate $\int_{0}^{\infty} e^{-3x^2} dx$. Recall that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$ for any value of $\mu$ and any $\sigma^2 > 0$.

**Homework Solutions:**

1. (a) $\int_{-\infty}^{\infty} f(y) dy = \int_{0}^{\frac{5}{25}} \frac{y}{25} dy + \int_{\frac{2}{25}}^{10} \left( \frac{2}{5} - \frac{y}{25} \right) dy = 1$
(b) \( P(Y \leq 3) = \int_0^3 \frac{y}{25} \, dy = \frac{9}{50} \)

(c) \( P(Y \leq 8) = \int_0^5 \frac{y}{25} \, dy + \int_5^8 \left( \frac{2}{5} - \frac{y}{25} \right) \, dy = .92 \)

(d) \( P(3 \leq Y \leq 8) = P(Y \leq 8) - P(Y \leq 3) = .74 \)

(e) \( P(Y \leq 2) + P(Y \geq 6) = \int_0^2 \frac{y}{25} \, dy + \int_6^{10} \left( \frac{2}{5} - \frac{y}{25} \right) \, dy = .4 \)

2. (a) \( P(X < 0) = F(0) = \frac{1}{2} \)
(b) \( P(-1 < X < 1) = F(1) - F(-1) = \frac{22}{32} \)
(c) \( P(5 < X) = 1 - P(X < .5) = 1 - F(.5) = .316 \)

3. (a) \( F(x) = \int_0^x \frac{3}{8} t^2 \, dt = \frac{x^3}{8} \)
(b) \( P(X \leq .5) = F(.5) = \frac{1}{16} \)
(c) \( P(.25 < X \leq .5) = F(.5) - F(.25) = .01367 \)
(d) \( E(X) = \int_0^2 \frac{3}{8} x^2 \, dx = \frac{3}{2} \)

\[ \text{Var}(X) = E(X^2) - E(X)^2 \text{ where } E(X^2) = \int_0^2 x^2 \frac{3}{8} x^2 \, dx = 2.4. \text{ Var}(X) = 2.4 - 1.5^2 = .15 \text{ and thus } \sigma_x = \sqrt{\text{Var}(X)} = .387. \]

4. (a) \( F(y) = \begin{cases} \frac{y^2}{50} & 0 \leq y < 5 \\ \frac{2y^2}{50} - \frac{y^2}{50} - 1 & 5 \leq y \leq 10 \\ 1 & 1 < y \end{cases} \) We get this using the following arguments:

- From 0 to 5, \( F(y) = \int_0^y \frac{t^2}{25} \, dt = \frac{y^3}{50} \)
- From 5 to 10, \( F(y) = \int_0^5 \frac{t^2}{25} \, dt + \int_5^y \left( \frac{2}{5} - \frac{t}{25} \right) \, dt = \frac{2y^3}{5} - \frac{y^2}{50} - 1 \)
- And of course \( F(y) = 1 \) for any value of \( y \) greater than or equal to 10.

5. \( E(Y) = \int_0^5 y \left( \frac{y}{25} \right) \, dy + \int_5^{10} y \left( \frac{2}{5} - \frac{y}{25} \right) \, dy = 5, \text{ and } E(Y^2) = \int_0^5 y^2 \left( \frac{y}{25} \right) \, dy + \int_5^{10} y^2 \left( \frac{2}{5} - \frac{y}{25} \right) \, dy = 29.1667. \) Thus \( \text{Var}(Y) = E(Y^2) - E(Y)^2 = 4.1667. \) If the time is uniformly distributed on \([0, 5]\), then the expected waiting time is 2.5 and the variance is \( \frac{25}{12} \).