Class # 27: Paired Comparisons

Reading: Section 9.3 of Devore
For the past several days, we have taken a look at how to compare two population means when the data collected from each population is independent. But what if the data are paired? In other words, let us assume that

\[ X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu_1, \sigma_1^2), \]

\[ Y_1, Y_2, \ldots, Y_n \overset{i.i.d.}{\sim} N(\mu_2, \sigma_2^2), \]

and that you were interested in the number \( \mu_1 - \mu_2 \). If the data were paired, i.e., if \( X_1 \) and \( Y_1 \) were observations made from the same experimental unit, then your methods would have to account for the fact that \( X_1 \) and \( Y_1 \) are correlated. If \( X_i \) and \( Y_i \) are correlated for each \( i \), and you were interested in gaining inference about the quantity \( \mu_1 - \mu_2 \), then it would be best to consider this sample of size \( 2n \) as a sample of \( n \) differences, \( d_1, d_2, \ldots, d_n \), where \( d_i = X_i - Y_i \). In this case,

\[ d_1, d_2, \ldots, d_n \overset{i.i.d.}{\sim} N(\mu_d, \sigma_d^2), \]

where \( \mu_d = \mu_1 - \mu_2 \), and \( \sigma_d^2 \) is the population variance of these paired differences. Inference about \( \mu_d = \mu_1 - \mu_2 \) could then be made using the traditional methods used before. In other words, if you wanted to test \( H_0 : \mu_d = \mu_1 - \mu_2 = \Delta_0 \) at level \( \alpha \) and \( \sigma_d^2 \) were known, that would merely involve calculating the \( z \)-score \( z = (\bar{d} - \Delta_0)/\sigma_d \) and then proceeding as outlined in Class #22 (Case 1). And if one wanted to calculate a 100\((1 - \alpha)\)% confidence interval (with \( \sigma_d^2 \) known), then that would be calculated as \( \bar{d} \pm z_{\frac{\alpha}{2}} \frac{\sigma_d}{\sqrt{n}} \). If \( \sigma_d^2 \) were not known, then the test \( H_0 : \mu_d = \Delta_0 \) at level \( \alpha \) would involve calculating the \( t \)-score \( t = (\bar{d} - \Delta_0)/\frac{s_d}{\sqrt{n}} \), where \( s_d = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (d_i - \bar{d})^2} \), and proceeding as outlined in Class #22 (Case 3). And a 100\((1 - \alpha)\)% confidence interval for \( \mu_d = \mu_1 - \mu_2 \) would then be calculated as \( \bar{d} \pm t_{\frac{\alpha}{2},n-1} \frac{s_d}{\sqrt{n}} \).

Homework: These problems are due at the beginning of class on 4/30/10.

1. Devore, page 341, #24. Note: In this case, SE mean = \( \frac{s_d}{\sqrt{n}} \).


5. Devore, page 360, #56.

Homework Solutions:

1. (a) In words, this \( H_a \) states that \( \mu_1 \), the average output of the subjects with the syndrome, is at least 1 unit less than \( \mu_2 \), the average heat output of the subjects without the syndrome. To test this, we calculate \( z = (.64 - 2.05 + 1)/\sqrt{\frac{.04}{10} + \frac{.16}{10}} = -2.899 \). The \( p \)-value is \( P(Z < -2.899) = \text{pnorm}(-2.899) = .00187 \) and since this is less than .01, we reject \( H_a \) in favor of \( H_o \).

(b) \[ P(\text{type II error}) = P(\text{failing to reject } H_o \text{ when } \mu_1 - \mu_2 = -1.2) \]

\[ = P \left\{ \frac{X_1 - X_2 + 1}{\sqrt{\frac{.04}{10} + \frac{.16}{10}}} > -2.32 \text{ if } \mu_1 - \mu_2 = -1.2 \right\} \]

\[ = P \left\{ \frac{X_1 - X_2 + 1.2}{\sqrt{\frac{.04}{10} + \frac{.16}{10}}} > -2.32 + \frac{1.2 - 1}{\sqrt{\frac{.04}{10} + \frac{.16}{10}}} \right\} \]

\[ = P(Z > -.906) = .817 \]
2. $H_0 : \mu_{men} - \mu_{women} = 0$ vs. $H_a : \mu_{men} - \mu_{women} > 0$ at $\alpha = .05$. In this case $z = (10.40 - 9.26)/\sqrt{\frac{2.83^2}{91} + \frac{4.68^2}{148}} = 1.828$. The $p$-value is $P(Z > 1.828) = .034$. Since $.034 < .05$, we reject $H_0$ in favor of $H_a$.

3. (a) For men: $19.39 \pm 1.96 \sqrt{\frac{2.52}{50}} = (18.7, 20.1)$. For women: $17.91 \pm 1.96 \sqrt{\frac{3.59}{50}} = (16.97, 18.85)$.

(b) This suggests that it might be the case that $\mu_{men} = \mu_{women}$, i.e., that there is no sex-based difference in manual agility.

(c) $19.39 - 17.91 \pm 1.96 \sqrt{\frac{2.52}{50} + \frac{3.59}{50}} = (.31, 2.65)$.

(d) This suggests that there is a sex-based difference in manual agility.

(e) The two-sample inference is the correct one.

(f) The results don’t agree for two important reasons: (i) The difference in the estimated standard deviations, and (ii) Doing the one-sample inference twice implies that there will be no difference with $.95 \times .95 = .9025$ confidence, but the two-sample inference implies that the difference is not 0 with 95% confidence.

4. $\mu_{highball} - \mu_{tumbler} = 0$ vs. $H_a : \mu_{highball} - \mu_{tumbler} \neq 0$. In this case $z = (42.2 - 60.9)/\sqrt{\frac{16.2^2}{90} + \frac{17.9^2}{90}} = -7.348$. The $p$-value is $P(Z < -7.348) + P(Z > 7.348) \approx 0$, so we would reject $H_0$ in favor of $H_a$ at $\alpha = .05$. 

