Fun with symplectic embeddings

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Outline

Hamiltonian mechanics

- Consider $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. This is the state space of a particle in $\mathbb{R}^n$, where the x_i are position coordinates and the y_i are momentum coordinates.
- Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function, which one can think of as a "Hamiltonian" or "energy" function. In classical mechanics, a particle has position coordinates x_i and momentum coordinates y_i depending on time t, evolving according to Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

That is, $(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))$ is a trajectory of the Hamiltonian vector field

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

Change of coordinates

- Suppose we change to new coordinates $\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n$. When will the equations of motion in the new coordinates be given by
$$\frac{d\hat{x}_i}{dt} = \frac{\partial H}{\partial \hat{y}_i}, \qquad \frac{d\hat{y}_i}{dt} = -\frac{\partial H}{\partial \hat{x}_i}?$$
- Define the standard symplectic form by
$$\omega = \sum_{i=1}^{n} dx_i\, dy_i.$$
The Hamiltonian vector field is characterized by
$$\omega(X_H, \cdot) = dH.$$
- It follows that a change of coordinates $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserves the equations of motion if $\varphi^*\omega = \omega$.

Symplectomorphisms and symplectic embeddings

Let U and V be domains in $\mathbb{R}^{2n}$.

Definition

- A symplectomorphism from U to V is a diffeomorphism $\varphi : U \to V$ such that $\varphi^*\omega = \omega$.
- A symplectic embedding of U into V is a smooth embedding $\varphi : U \to V$ such that $\varphi^*\omega = \omega$.

Example

If $n = 1$, then a symplectic embedding is just an area-preserving smooth embedding.

Symplectic versus volume-preserving

Remark

For any n, if there exists a symplectic embedding $\varphi : U \to V$, then $\mathrm{vol}(U) \le \mathrm{vol}(V)$.

Proof. Observe that

$$\omega^n = \frac{1}{n!} dx_1\, dy_1 \cdots dx_n\, dy_n.$$

Thus φ pulls back the volume form to the volume form. So

$$\mathrm{vol}(U) = \mathrm{vol}(\varphi(U)) \le \mathrm{vol}(V).$$

Question

Is there any significant difference between symplectic embeddings and volume-preserving embeddings?

The Gromov nonsqueezing theorem

Notation:

- Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ with coordinates $z_i = x_i + \sqrt{-1}y_i$.
- For $a > 0$, define the ball

$$B(a) = \left\{ z \in \mathbb{C}^n \,\middle|\, \pi|z|^2 \le a \right\}.$$

Define the cylinder

$$Z(a) = \left\{ z \in \mathbb{C}^n \,\middle|\, \pi|z_1|^2 \le a \right\}.$$

Theorem (Gromov, 1985)

Suppose there exists a symplectic embedding $\varphi : B(r) \to Z(R)$. Then $r \le R$.

Note that there are volume-preserving embeddings $B(r) \to Z(R)$ for $r > R$. (One can use a diagonal linear map.)

Outline of proof of Gromov nonsqueezing

Suppose there exists a symplectic embedding $B(r) \to Z(R)$. Let $\varepsilon > 0$. One uses the theory of pseudoholomorphic curves to produce a surface $\Sigma \subset B(r)$ such that:

(a) $0 \in \Sigma$.

(b) $\partial\Sigma \subset \partial B(r)$.

(c) The tangent space to Σ at each point is invariant under multiplication by $\sqrt{-1}$.

(d) $\mathrm{Area}(\Sigma) < R + \varepsilon$.

Condition (c) and Wirtinger's inequality imply that Σ is *area-minimizing* (relative to its boundary). Conditions (a) and (b) and the monotonicity lemma for minimal surfaces then imply that

$$r \leq \mathrm{Area}(\Sigma).$$

By condition (d), we get $r < R + \varepsilon$. Since ε was arbitrary, we conclude that $r \leq R$.

Gromov nonsqueezing and the uncertainty principle

Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}^2$ denote the projection on to the x_1, y_1 plane.

Theorem (equivalent formulation of Gromov nonsqueezing)

If $\varphi : B(r) \to \mathbb{R}^{2n}$ is a symplectic embedding, then

$$\mathrm{Area}(\rho(\varphi(B(r)))) \geq r.$$

- If one has an unknown point in the ball $B(r)$, then there is some uncertainty in the values of the position x_1 and the momentum y_1. The theorem asserts that one cannot reduce this uncertainty (as measured by the area of the set of possible values of x_1 and y_1) by a symplectic change of coordinates.
- Thus, one can regard Gromov nonsqueezing as a classical analogue of the Heisenberg uncertainty principle in quantum mechanics.

More general symplectic embedding questions

General question

Given domains $U, V \subset \mathbb{R}^{2n}$, when does there exist a symplectic embedding $U \to V$?

This is hard even for simple examples such as:

Definition

Let $a_1, \ldots, a_n > 0$. Define the ellipsoid

$$E(a_1, \ldots, a_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \leq 1 \right\}.$$

Define the polydisk

$$P(a_1, \ldots, a_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi |z_i|^2 \leq a_i, \;\; \forall i = 1, \ldots, n \right\}.$$

Four-dimensional ellipsoids

Theorem (McDuff, 2010)

$\mathrm{int}(E(a,b))$ *symplectically embeds into* $E(c,d)$ *if and only if*

$$N(a,b) \le N(c,d).$$

- Here $N(a,b)$ denotes the sequence of all nonnegative integer linear combinations of a and b, written in increasing order with repetitions. For example,

$$N(1,1) = (0,1,\color{red}{1},2,2,2,3,3,3,3,\ldots),$$
$$N(1,2) = (0,1,\color{red}{2},2,3,3,4,4,4,5,5,5,\ldots).$$

  It follows that $E(1,2)$ symplectically embeds into $E(a,a) = B(a)$ if and only if $a \ge 2$.
- Given arbitrary a, b, c, d, it is nontrivial to decide whether $N(a,b) \le N(c,d)$.
- The higher dimensional analogue of the "only if" part of this theorem is false (Guth, Hind-Kerman).

Embedding a 4d ellipsoid into a ball

Given $a \geq 1$, define $f(a)$ to be the infimum over c such that $E(a,1)$ symplectically embeds into $B(c)$.

Theorem (McDuff-Schlenk)

- *For* $1 \leq a \leq \left(\frac{1+\sqrt{5}}{2}\right)^4$*, the function* f *is piecewise linear, described explicitly by a "Fibonacci staircase".*
- *The interval* $\left[\left(\frac{1+\sqrt{5}}{2}\right)^4, (17/6)^2\right]$ *is divided into finitely many subintervals, on each of which either* f *is linear or* $f(a) = \sqrt{a}$*.*
- *If* $a \geq (17/6)^2$*, then* $f(a) = \sqrt{a}$*.*

Note that $f(a) \geq \sqrt{a}$ is the volume constraint, because $\mathrm{vol}(E(a,1)) = a/2$ and $\mathrm{vol}(B(c)) = c^2/2$.

Embedding a 4d polydisk into a ball

For $a \geq 1$, define $g(a)$ to be the infimum over c such that $P(a,1)$ symplectically embeds into $B(c)$. The obvious inclusion $P(a,1) \subset B(a+1)$ shows that $g(a) \leq a+1$. When $a > 2$, "symplectic folding" (due to Schlenk) can be used to show that $g(a) \leq 2 + a/2$.

Theorem (H., 2014)

- *If* $1 \leq a \leq 2$ *then* $g(a) = a + 1$*.*
- *If* $2 \leq a \leq 12/5$ *then* $g(a) = 2 + a/2$*.*

- The method of proof may be able to improve $12/5$ to $(\sqrt{7}-1)/(\sqrt{7}-2)$.
- Hind-Lisi previously showed that $g(2) = 3$.

Embedding a 4d polydisk into a cube

Theorem (H., 2014)

If $1 \le a \le 2$, then $P(a,1)$ symplectically embeds into $P(c,c)$ if and only if $a \le c$.

- If $a \le c$ then there is an obvious inclusion $P(a,1) \to P(c,c)$.
- If $a > 2$ then one can do better: one can use symplectic folding to show that $P(a,1)$ symplectically embeds into $P(c,c)$ whenever $c > 1 + a/2$.

Symplectic capacities

Definition

A symplectic capacity (for domains in $\mathbb{R}^{2n}$) is a function c, associating to each domain U a real number $c(U) \geq 0$, such that:

- If there exists a symplectic embedding $U \to V$, then $c(U) \leq c(V)$.
- if $r > 0$ is a constant then $c(rU) = r^2 c(U)$.

Example

The Gromov width of U is defined by

$$c_{\mathrm{Gr}}(U) = \sup \{a \mid \exists \text{ symp. emb. } B(a) \to U\}.$$

Example

Define the uncertainty of U by

$$c_{\mathrm{unc}}(U) = \inf \{a \mid \exists \text{ symp. emb. } U \to Z(a)\}.$$

Viterbo's conjecture

- Gromov nonsqueezing is equivalent to $c_{\mathrm{Gr}}(Z(a)) = a$, and also to $c_{\mathrm{unc}}(B(a)) = a$.
- It follows from the definitions that if c is any symplectic capacity with $c(B(a)) = c(Z(a)) = a$ and U is any domain then

$$c_{\mathrm{Gr}}(U) \le c(U) \le c_{\mathrm{unc}}(U).$$

Viterbo's conjecture

If U is a *convex* domain then $c_{\mathrm{Gr}}(U) = c_{\mathrm{unc}}(U)$. Equivalently, all capacities c such that $c(B(a)) = c(Z(a)) = a$ agree on all convex sets.

- Ostrover et al have shown that Viterbo's conjecture implies the Mahler conjecture in convex geometry.
- The conjecture becomes false if one allows nonconvex sets.

More symplectic capacities

- The Gromov width c_{Gr} and the uncertainty c_{unc} are easy to define, but hard to compute.
- One can define other, more computable symplectic capacities, using various flavors of "contact homology".
- For example, one can use "embedded contact homology" to define the ECH capacities of domains $U \subset \mathbb{R}^4$.

The ECH capacities of a domain $U \subset \mathbb{R}^4$ are a sequence of real numbers

$$0 = c_0(U) \le c_1(U) \le c_2(U) \le \cdots \le \infty.$$

Some properties of ECH capacities

- If there exists a symplectic embedding $U \to V$, then $c_k(U) \le c_k(V)$ for all k.
- The ECH capacities of an ellipsoid are given by

$$(c_k(E(a,b)))_{k=0,1,\ldots} = N(a,b).$$

- The ECH capacities of a polydisk are

$$c_k(P(a,b)) = \min\{am + bn \mid m,n \in \mathbb{N},\ (m+1)(n+1) \ge k+1\}.$$

- If $U_1, \ldots, U_m$ are disjoint, then

$$c_k\left(\coprod_{i=1}^m U_i\right) = \max_{k_1+\cdots+k_m=k} \sum_{i=1}^m c_{k_i}(U_i).$$

- Asymptotics and volume:

$$\lim_{k\to\infty} \frac{c_k(U)^2}{k} = 4\,\mathrm{vol}(U).$$

Example: the volume property for the ellipsoid

Let's check that

$$\lim_{k\to\infty} \frac{c_k(E(a,b))^2}{k} = 4\,\mathrm{vol}(E(a,b)).$$

- Let T_L denote the triangle bounded by the the x and the y axes and the line $ax + by = L$.
- $c_k(E(a,b))$ is the smallest real number L such that the triangle T_L contains at least $k+1$ lattice points.
- Thus $c_k(E(a,b)) = L$ implies $\mathrm{Area}(T_L) \approx k$.
- But
$$\mathrm{Area}(T_L) = \frac{L^2}{2ab}.$$
- Thus
$$\lim_{k\to\infty} \frac{c_k(E(a,b))^2}{k} = 2ab.$$
- Since $\mathrm{vol}(E(a,b)) = ab/2$, it works.

The Reeb vector field

Let U be a star-shaped domain in $\mathbb{R}^{2n}$ with smooth boundary Y. "Star-shaped" here means that ∂U is transverse to the radial vector field. Also, let

$$\lambda = \frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i - y_i\,dx_i).$$

Definition

The Reeb vector field on Y is the unique vector field R such that:

- $\omega(R, v) = 0$ for all v tangent to Y.
- $\lambda(R) = 1$.

If $H : \mathbb{R}^{2n} \to \mathbb{R}$ is any smooth function having Y as a regular level set, then R is a rescaling of the Hamiltonian vector field X_H on Y.

Definition

A Reeb orbit is a periodic orbit γ of R.

Some conjectures about Reeb dynamics

Old conjecture

For any compact star-shaped domain in $\mathbb{R}^{2n}$, the boundary has at least n Reeb orbits. (Known for $n = 2$.)

If γ is a Reeb orbit, define its symplectic action $A(\gamma) > 0$ to be its period. Equivalently,

$$A(\gamma) = \int_\gamma \lambda = \int_D \omega$$

where D is a smooth disk with boundary ω.

Conjecture (consequence of Viterbo's conjecture)

If U is a convex *compact star-shaped domain, then there is a Reeb orbit γ on ∂U with*

$$A(\gamma)^n \leq n!\,\mathrm{vol}(U).$$

Reebb dynamics and symplectic capacities

Fact (Hofer et al)

There is a symplectic capacity c (the first Ekeland-Hofer capacity) such that if U is compact and convex, then $c(U)$ is the minimal symplectic action of a Reeb orbit on ∂U.

- More generally, most computable symplectic capacities are defined in terms of symplectic actions of Reeb orbits on the boundary.
- In particular, the k^{th} ECH capacity $c_k(U)$ is a positive integer linear combination of symplectic actions of Reeb orbits on ∂U.
- Monotonicity under symplectic embeddings holds because pseudoholomorphic curves interpolating between Reeb orbits have positive symplectic area.

Reeb dynamics on the boundary of an ellipsoid

- Recall the 4d ellipsoid

$$E(a,b) = \left\{ z \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1 \right\}.$$

- In polar coordinates $z_i = r_i e^{\sqrt{-1}\theta_i}$, the Reeb vector field on $\partial E(a,b)$ is

$$R = \frac{2\pi}{a}\frac{\partial}{\partial\theta_1} + \frac{2\pi}{b}\frac{\partial}{\partial\theta_2}.$$

- Thus $\gamma_1 = (z_2 = 0)$ is a Reeb orbit of symplectic action a, and $\gamma_2 = (z_1 = 0)$ is a Reeb orbit of symplectic action b.
- If a/b is irrational, then there are no other Reeb orbits. Thus the ECH capacities are *all* of the positive integer linear combinations of symplectic actions of Reeb orbits.
- For more general domains, ECH capacities are certain *distinguished* linear combinations of symplectic actions, selected homologically.

Conclusion

- Already in the four-dimensional case, there is much work to do to compute symplectic capacities and understand when symplectic embeddings are possible, with mysterious connections to number theory and combinatorics. There are many projects suitable for undergraduate research.
- For more information see *Beyond ECH capacities*, arXiv:1409.1352, and the references therein.