The Isoperimetric Problem in $\mathbb{R}^n$ with Density r^p

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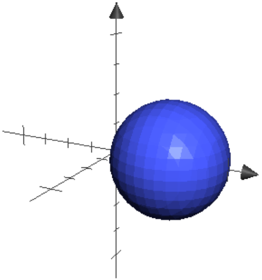
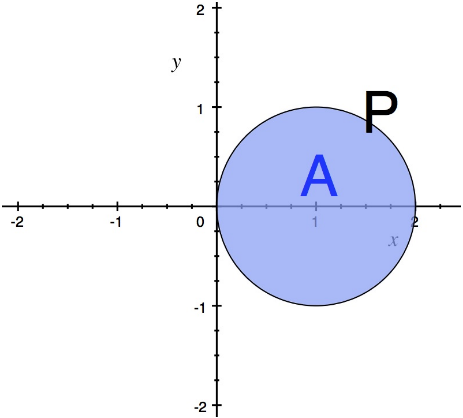


Williams College NSF SMALL Undergraduate Research Geometry Group

Wyatt Boyer, Bryan Brown, Alyssa Loving, Sarah Tammen

2D

Theorem [G 2008] In $\mathbb{R}^2$ with density r^p ($p > 0$), for a given weighted area A, the least weighted perimeter P is given by a circle through origin.

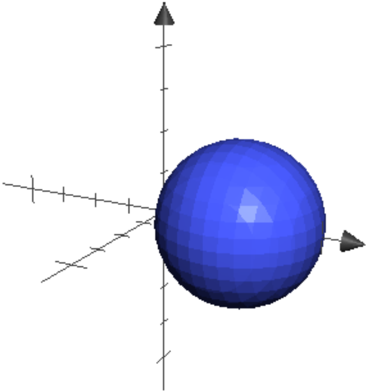


2008 Geometry Group: Jon Dahlberg, Alex Dubbs, Ed Newkirk, Hung Tran

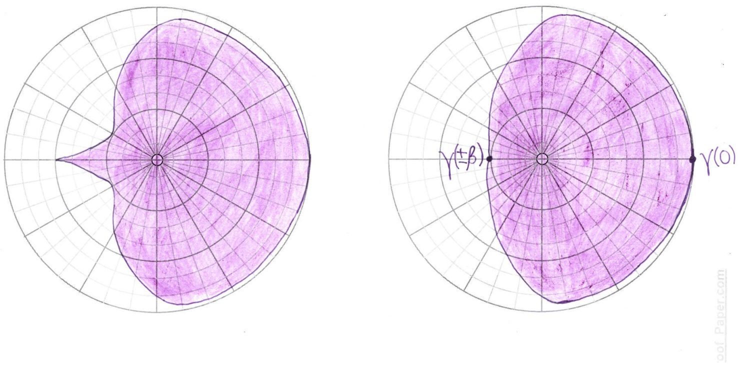
Conj. [G 2009] In $\mathbb{R}^n$ with density r^p $(p > 0)$, spheres through the origin are uniquely isoperimetric.

Proof - Gregory Chambers and 2014 Geometry Group, after Chambers' proof of Log Convex Density Conjecture

2009 Geometry Group: Alex Díaz, Nate Harman, Sean Howe, David Thompson



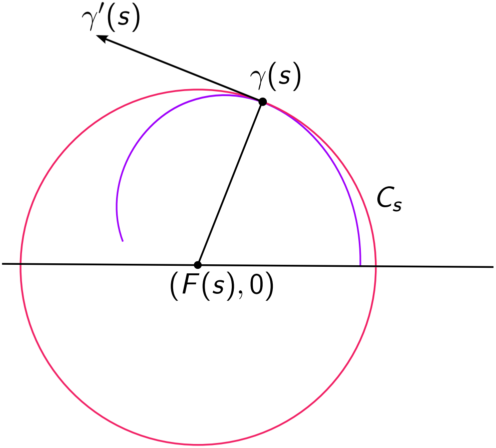
Spherical Symmetrization preserves weighted volume but reduces weighted perimeter.



On upper half of curve, $|\gamma|$ not increasing

Canonical Circle

- Canonical circle C_s is tangent to γ at $\gamma(s)$ and has center on x-axis.
- $\lambda(s) :=$ curvature of C_s
- $\lambda(s) = \kappa(s)$ & C_s goes through origin $\implies$ γ is circle through origin.



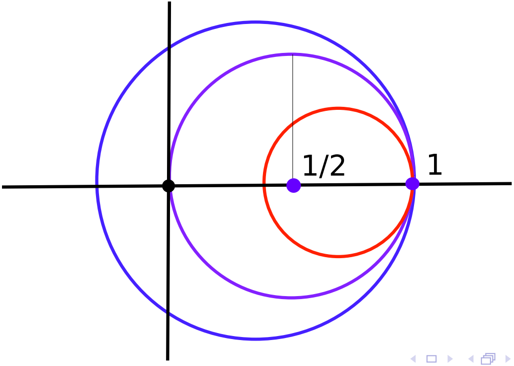
Basis for Cases

$\kappa(0) = \lambda(0)$, so if C_0 goes through origin, then γ is circle through origin.

Note that C_0 goes through origin $\iff$ center at $1/2$.

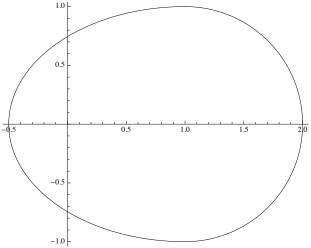
Proof by contradiction - take two cases:

- right case: Center of C_0 is right of $1/2$.
- left case: Center of C_0 is left of $1/2$.

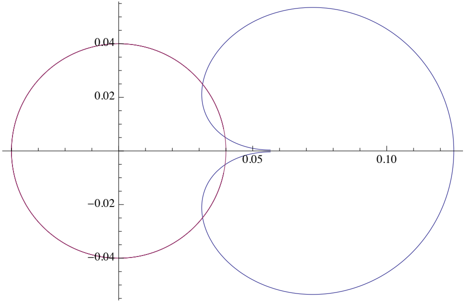


2 Cases

Left Case?



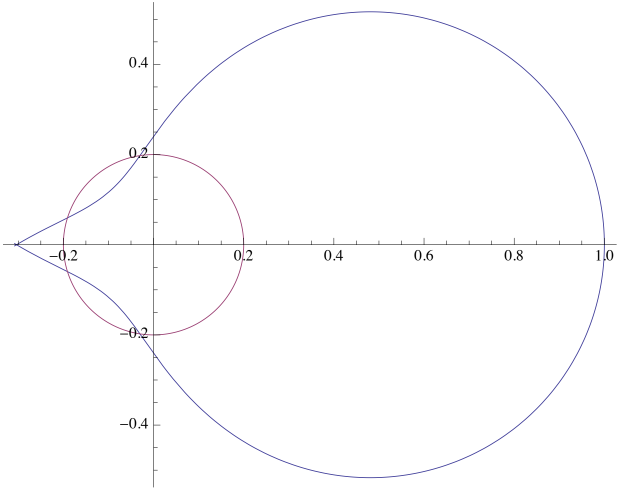
Right Case



Violates Spherical Symmetry

Left Case

Left case also violates spherical symmetry.



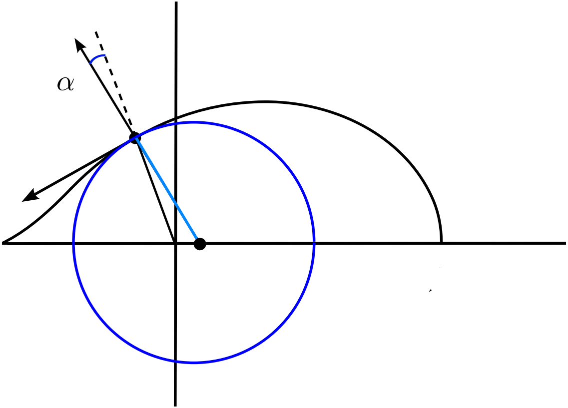
Proof, Left Case - Generalized Mean Curvature

For radial density $f(x) = e^{g(x)}$, generalized mean curvature defined by $H_f(x) = H_0(x) + \frac{\partial g}{\partial \nu}(x)$.

For density r^p,
$g(x) = log(|x|^p)$, and

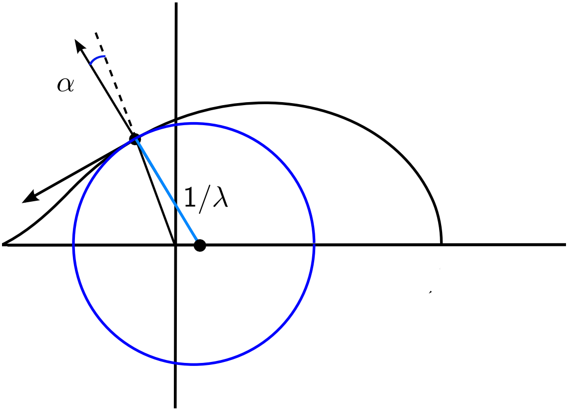
$$\frac{\partial g}{\partial \nu} = p\,\frac{x}{|x|^2} \cdot \nu(x).$$

We define $\mu = (p/r)\cos\alpha$, where α is angle between position vector and unit outward normal.



Proof, Left Case

const $=$ generalized mean curvature $= \kappa + (n-2)\lambda + \mu$,
where $\mu = (p/r)\cos\alpha$



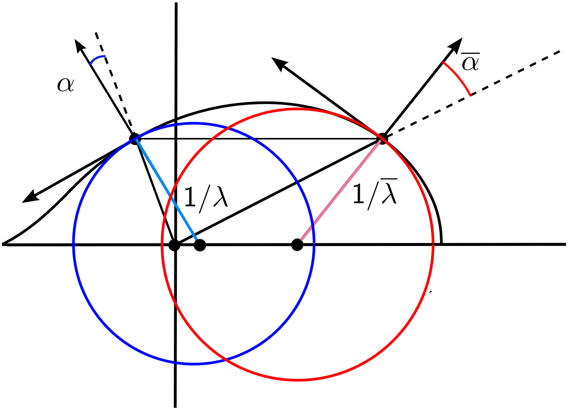
Proof, Left Case (cont'd)

const $=$ generalized mean curvature $= \kappa + (n-2)\lambda + \mu$,
where $\mu = (p/r)\cos\alpha$.

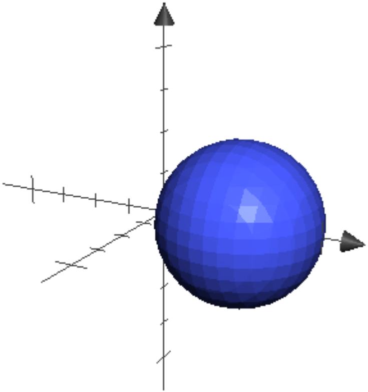
$\lambda \geq \overline{\lambda}$,
$\mu > \overline{\mu} \implies \kappa < \overline{\kappa}$

Check at top and show that inequalities continue to hold.

Now γ meets at angle $< 90^\circ$.



Thm. [G 2014] In $\mathbb{R}^n$ with density r^p $(p > 0)$, spheres through the origin are uniquely isoperimetric.



Thank You.