Comparison Geometry for Manifolds with Density

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Manifold with Density

A manifold with density is a Riemannian manifold (M, g) equipped with a positive density function, e^{-f}.

For example, consider a surface embedded in $\mathbb{R}^3$, made of a material of variable density e^{-f}.

"Perelmans proof of the Poincaré Conjecture requires placing a positive, continuous "density" function on the manifold. Manifolds with density appear a number of places in mathematics ....The grand goal is to generalize all of Riemannian geometry to manifolds with density."

- Frank Morgan

Levi-Civita Connection

Fundamental Thm of Riemannian Geometry On any Riemannian manifold (M, g) there is a unique smooth affine connection, ∇ which is torsion free and compatible with g.

Let $\phi : M \to \mathbb{R}$, define

$$\nabla^{d\phi}_X Y = \nabla_X Y - d\phi(X)Y - d\phi(Y)X.$$

$\nabla^{d\phi}$ is a torsion free affine connection which is *projectively equivalent* to ∇, i.e. it has the same geodesics up to re-parametrization. (Weyl '21).

Let $\gamma(r)$ a geodesic for ∇, then $s(r) = \int_0^r e^{-2\phi} dt$ is the parameter for a $\nabla^{d\phi}$-geodesic, $\widetilde{\gamma}(s)$.

Curvature

A connection is sufficient to define a curvature tensor and a Ricci tensor.

- $R^{d\phi}(X,Y)Z = \nabla^{d\phi}_X \nabla^{d\phi}_Y Z - \nabla^{d\phi}_Y \nabla^{d\phi}_X Z - \nabla^{d\phi}_{[X,Y]} Z$
- $\mathrm{Ric}^{d\phi}(Y,Z) = \mathrm{tr}(X \to R^{d\phi}(X,Y)Z)$.

The N-Bakry-Émery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{df \otimes df}{N - \dim(M)}.$$

When $N = \infty$ we write $\mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f$.

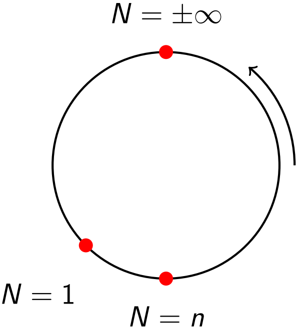
(W.-Yeroshkin) Suppose (M^n, g) is a Riemannian manifold with $n > 1$ and $f \in C^2(M, \mathbb{R})$. If $\phi = \frac{f}{n-1}$, then $\mathrm{Ric}^{d\phi} = \mathrm{Ric}_f^1$.

"Negative" synthetic dimension

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{df \otimes df}{N - \dim(M)}.$$

N is called the "synthetic dimension" parameter. Traditionally N is assumed to be $N > \dim(M)$ or $N = \infty$ but it also makes perfect sense when $N < \dim(M)$.

The conditions $\mathrm{Ric}_f^N \geq \lambda g$, $N < 0$ and $N < 1$ have been studied recently in (Ohta '13, Kolesnikov-Milman '13, Milman, '14).



As N increases, $\mathrm{Ric}_f^N \geq \lambda g$ becomes a weaker condition.

Comparison Geometry

Bigger curvature $\rightsquigarrow$ smaller manifold.

Myers' Theorem Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)Kg$, $K > 0$, then $\mathrm{diam} \leq \frac{\pi}{\sqrt{K}}$. In particular, M is compact and $\pi_1(M) < \infty$.

(Qian, '97) Myers' Theorem holds when $\mathrm{Ric}_f^N \geq (N-1)Kg$, $K > 0$ and $N > n$.

Gauss Space $(\mathbb{R}^n, g_{flat})$ with $f(x) = \frac{1}{2}|x|^2$ has $\mathrm{Ric}_f^\infty = g$ but M is noncompact.

(Morgan, '05) Let (M^n, g) be a complete Riemannian manifold supporting a function f with $\mathrm{Ric}_f^\infty \geq \lambda g$, $\lambda > 0$, then $\int_M e^{-f} dvol_g < \infty$. In particular, $\pi_1(M) < \infty$.

Myers' Theorem for Ric_f^1?

For all manifolds N, $\mathbb{R} \times N$ supports and Riemannian metric and density such that $\mathrm{Ric}_f^1 \geq Kg$, $K > 0$.

Let $\gamma(r)$ be a unit speed geodesic for g.

$s(r) = \int_0^r e^{\frac{-2(f\circ\gamma)(t)}{n-1}} dt$ is the parameter for a $\nabla^{\frac{df}{n-1}}$-geodesic, $\widetilde{\gamma}(s)$.

Therefore,

$$\mathrm{Ric}^{\frac{df}{n-1}}\left(\frac{d\widetilde{\gamma}}{ds}, \frac{d\widetilde{\gamma}}{ds}\right) \geq (n-1)K \quad \Leftrightarrow \quad \mathrm{Ric}_f^1\left(\frac{d\gamma}{dr}, \frac{d\gamma}{dr}\right) \geq (n-1)Ke^{\frac{-4f}{n-1}}.$$

Thm A (W.-Yeroshkin) Let (M, g) is a complete Riemannian manifold. If there is a function f such that $\nabla^{\frac{df}{n-1}}$ is geodesically complete and $\mathrm{Ric}_f^1 \geq (n-1)Ke^{\frac{-4f}{n-1}}g$, $K > 0$ then (M, g) is compact.

A connection is called *geodesically complete* if the geodesics exist for all time.

Corollary Let (M^n, g), is a complete Riemannian manifold such that $\mathrm{Ric}_f^1 \geq \lambda g$, $\lambda > 0$ and $|f| \leq k$ then M is compact.

Improves result of (Wei-W., '09) for $\mathrm{Ric}_f^\infty \geq \lambda g$, $|f| \leq k$.

If M is orientable then $e^{-\frac{n+1}{n-1}f} dvol_g$ is parallel with respect to $\nabla^{\frac{df}{n-1}}$.

Thm B (W.-Yeroshkin) Let (M^n, g), is a complete Riemannian manifold such that $\mathrm{Ric}_f^1 \geq (n-1)Ke^{\frac{-4f}{n-1}} g$, $K > 0$ then $\int_M e^{-\frac{n+1}{n-1}f} dvol_g < \infty$. In particular, $\pi_1(M) < \infty$.

We also obtain versions of Lichnerowicz-Cheeger-Gromoll Splitting Theorem, Bishop-Gromov volume comparison, Laplacian comparison, Cheng's maximal diameter theorem for Ric_f^1.

In particular, we obtain results for $\mathrm{Ric}_f^1 \geq \lambda g$, $|f| \leq k$ that were previously only known for $\mathrm{Ric}_f^\infty \geq \lambda g$.

Other Structures coming from connection

(M,g) Riemannian manifold, $\nabla^{d\phi}$ weighted connection.

Weighted Sectional Curvature

Let U, V be an orthonormal pair of vectors,

$$g(R^{d\phi}(V,U)U,V) = \overline{sec}_{\phi}^{U}(V) = \mathrm{sec}(U,V) + \mathrm{Hess}\phi(U,U) + g(\nabla\phi, U)^2.$$

Studied earlier in (W. '13, Kennard-W, '14).

New Splitting Theorem (W.-Yeroshkin) Suppose (M, g) is a simply connected, complete Riemannian manifold. If $\nabla^{d\phi}$ admits k linearly independent parallel vector fields, then M splits as one of the following:

$$M = \mathbb{R}^k \times N \qquad g_M = g_{Eucl} + e^{2\psi} g_N \qquad \phi = \psi_{Eucl} + \phi_N$$

$$M = \mathbb{H}^k \times N \qquad g_M = g_{Hyp} + e^{2\psi} g_N \qquad \phi = \psi_{Hyp} + \phi_N$$

Thank you for your attention!

References for Ric_f^N, $N < \dim(M)$

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