ISOPERIMETRIC PROBLEMS IN SECTORS WITH DENSITY - REPORT 105 OF 7/27

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Abstract. We consider the isoperimetric problem in planar sectors with density r^p, and with density $a > 1$ inside the unit disk and 1 outside. We characterize solutions as a function of sector angle. We provide a very general symmetrization theorem, and apply it to $\mathbb{R}^n$ with radial density.

Contents

1. Introduction

A *density* is a function weighting both perimeter and area. We study the isoperimetric problem on planar sectors with certain densities. The isoperimetric problem seeks to enclose prescribed (weighted) area with least (weighted) perimeter. Solutions are known for very few surfaces with densities (see Sect. 2 below). Our first major theorem after Dahlberg *et al.* [DDNT, Thm. 3.16] characterizes isoperimetric curves in a θ_0-sector with density r^p, $p > 0$:

Theorem. *(5.19) Given $p > 0$, there exist $0 < \theta_1 < \theta_2 < \infty$ such that in the θ_0-sector with density r^p, isoperimetric curves are (see Fig. 1.1):*
1. *for $0 < \theta_0 < \theta_1$, circular arcs about the origin,*
2. *for $\theta_1 < \theta_0 < \theta_2$, unduloids,*
3. *for $\theta_2 < \theta_0 < \infty$, semicircles through the origin.*

We give bounds on θ_1 and θ_2, but are unable to determine them exactly. Section 6 gives further results on constant generalized curvature curves. Sectors with density r^p are related to L^p spaces (see *e.g.* Cor. 5.27), have vanishing generalized Gauss curvature [CHHSX, Def. 5.1], and have an interesting singularity at the origin where density vanishes. Adams *et al.* [ACDLV] previously studied sectors with Gaussian density.

Our second major theorem after Cañete *et al.* [CMV, Thm. 3.20] characterizes isoperimetric curves in a θ_0-sector with density $a > 1$ inside the unit disk and

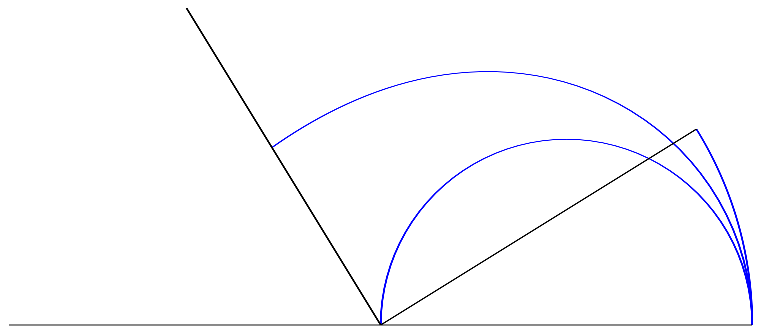


FIGURE 1.1. The three minimizers for sectors with density r^p: a circular arc about the origin for small sectors, an unduloid for medium sectors, and a semicircle through the origin for large sectors.

density 1 outside the unit disk. An interesting property of this problem is that it deals with a noncontinuous density. There are five different kinds of minimizers depending on θ_0, a, and the prescribed area, shown in Figure 7.1.

Section 8 provides a general symmetrization theorem, including Steiner, Schwarz, and spherical symmetrization in products, warped products, and certain fiber bundles. Section 9 provides applications to $\mathbb{R}^n$ with radial densities.

1.1. **The Sector with Density** r^p**.** In the plane with density r^p, Carroll *et al.* [CJQW, Sect. 4] prove that for $p < -2$, isoperimetric curves are circles about the origin bounding area on the outside, and prove that for $-2 \leq p < 0$, isoperimetric regions do not exist. Dahlberg *et al.* [DDNT, Thm. 3.16] prove that for $p > 0$, isoperimetric curves are circles through the origin. By a simple symmetry argument (Prop. 4.2), isoperimetric circles about the origin and circles through the origin in the plane correspond to isoperimetric circular arcs about the origin and semicircles through the origin in a π-sector. In this paper, we consider θ_0-sectors for general $0 < \theta_0 < \infty$.

For $p \in (-\infty, -2) \cup (0, \infty)$, existence in the θ_0-sector follows from standard compactness arguments (Prop. 3.11). Lemma 5.6 limits the possibilities to circular arcs about the origin, semicircles through the origin, and unduloids (nonconstant positive polar graphs with constant generalized curvature). Proposition 5.2 shows that if the circle is not uniquely isoperimetric for some angle θ_0, it is not isoperimetric for all $\theta > \theta_0$. Corollary 5.14 shows that if the semicircle is ever minimizing, it is uniquely minimizing for all angles greater. Therefore, transitional angles $0 \leq \theta_1 \leq \theta_2 \leq \infty$ exist. Minimizers that depend on sector angle have been seen before, as in the characterization by Lopez and Baker [LB, Thm. 6.1, Fig. 10] of perimeter-minimizing double bubbles in the Euclidean cone of varying angles, which is equivalent to the Euclidean sector. Theorem 5.19 provides estimates on the value of θ_1 and θ_2.

We conjecture (Conj. 5.20) that $\theta_1 = \pi/\sqrt{p+1}$ and $\theta_2 = \pi(p+2)/(2p+2)$. Proposition 5.18 proves that the circle about the origin has positive second variation for all $\theta_0 < \pi/\sqrt{p+1}$, and Proposition 5.10 proves the semicircle through the origin

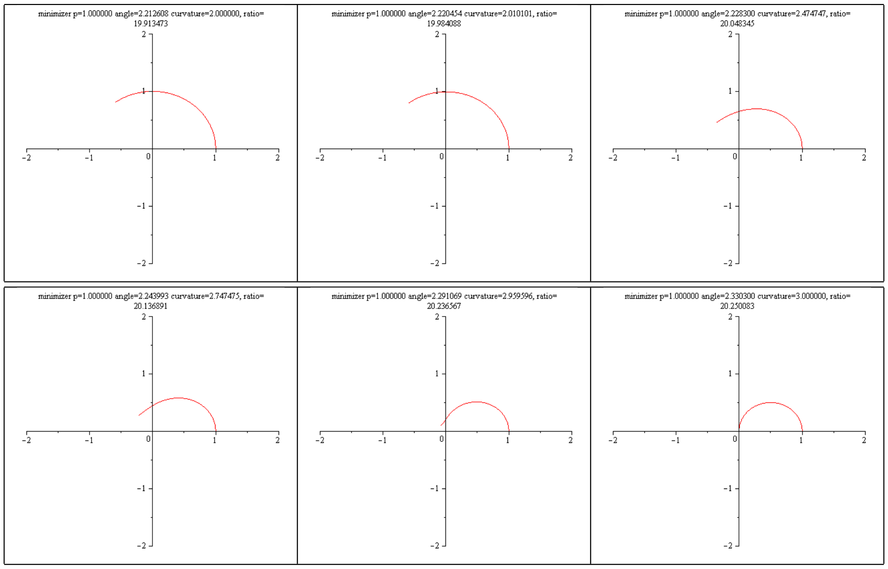


FIGURE 1.2. Using Maple we can predict the transition of minimizers from the circular arc through unduloids to the semicircle as per Conjecture 5.20 as can be seen above for $p = 1$.

is not isoperimetric for all $\theta_0 < \pi(p+2)/(2p+2)$. Numerics (Fig. 1.2) also support our conjecture.

An easy symmetrization argument (Prop. 4.2) shows that the isoperimetric problem in the θ_0-sector is equivalent to the isoperimetric problem in the $2\theta_0$-cone, a cone over $\mathbb{S}^1$. We further note that the isoperimetric problem in the cone over $\mathbb{S}^1$ with density r^p is equivalent to the isoperimetric problem in the cone over the product of $\mathbb{S}^1$ with a p-dimensional manifold M among regions symmetric under a group of isometries acting transitively on M. This provides a classical interpretation of the problem, which we use in Proposition 5.9 to obtain an improved bound for θ_1 in the $p = 1$ case by taking M to be the rectangular two-torus.

1.2. **Constant Generalized Curvature Curves.** Section 6 provides further details on constant generalized curvature curves in the sector with density r^p, which are of interest since minimizers must have constant generalized curvature (see Sect. 2). Proposition 6.14 shows that if the unduloid periods are bounded above and below by certain values, then we can determine the exact values of θ_1 and θ_2 (see Conj. 5.20).

1.3. **The Sector with Disk Density.** Section 7 considers a sector of the plane with density $a > 1$ inside the unit disk and 1 outside. Cañete *et al.* [CMV, Sect. 3.3] consider this problem in the plane, which is equivalent to the π-sector. Proposition 7.2 gives the five possibilities of Figure 7.1. Our Theorems 7.8, 7.9, and 7.10 classify isoperimetric curves in a θ_0-sector, depending on θ_0, density a, and area.

1.4. **Symmetrization.** In section 8, we provide two general symmetrization theorems in arbitrary dimension and codimension, in products, warped products, and certain fiber bundles, including Steiner, Schwarz, and spherical symmetrization.

Proposition 8.3 extends a symmetrization theorem of Ros [R1, Sect. 3.2] to warped products with a product density, and is general enough to include spherical symmetrization as well as Steiner and Schwarz symmetrization. Proposition 8.6 extends symmetrization to Riemannian fiber bundles with equidistant Euclidean fibers such as certain lens spaces.

1.5. $\mathbb{R}^n$ **with Radial Density.** Section 9 considers the isoperimetric problem in $\mathbb{R}^n$ with radial density. We use spherical symmetrization to reduce the problem to a two dimensional isoperimetric problem in a plane with density. We specifically consider $\mathbb{R}^n$ with density r^p and provide a conjecture (Conj. 9.3) and a nonexistence result (Prop. 9.5).

1.6. **Computations.** Section 10 discusses the Maple program we used to produce numerical estimates for the minimizers. As depicted in Figure 1.2, the program predicts that for a given value of p, isoperimetric curves start as circular arcs for small sector angles and then transition smoothly to the semicircle.

1.7. **Open Questions.**

(1) How can one prove Conjecture 5.20 on the values of the transitional angles θ_1 and θ_2?
(2) Could the values of θ_1 and θ_2 be proven numerically for fixed p?
(3) If a circular arc about the origin is isoperimetric in the θ_0-sector with density r^p, is it isoperimetric in the same θ_0-sector with density r^q, $q < p$?
(4) Are circles about the origin isoperimetric in the Euclidean plane with perimeter density r^p, $p \in (0, 1)$? (See Rmk. after Conj. 5.25.)
(5) In the θ_0-sector with density r^p, do curves with constant generalized curvature near that of the semicircle have half period $T \approx \pi\,(p + 2)\,/\,(2p + 2)$? (See Conj. 5.20 and Prop. 6.12 for the corresponding result near the circular arc.)
(6) Are spheres through the origin isoperimetric in $\mathbb{R}^n$ with density r^p, $p > 0$? (See Conj. 9.3.)

1.8. **Acknowledgments.** This paper is the work of the 2009 SMALL Geometry Group in Granada, Spain. We would like to thank the National Science Foundation and Williams College for funding both SMALL and our trip to Granada. We would also like to thank the University of Granada's Department of Geometry and Topology for their work and support. In particular, we thank Rafael López for opening his home and his office to us. We would like to thank Antonio Cañete, Manuel Ritoré, Antonio Ros, and Francisco López for their comments. We thank César Rosales for arranging our wonderful accommodations at Carmen de la Victoria. We thank organizers Antonio Martínez, José Antonio Gálvez, and Francisco Torralbo of the Escuela de Análisis Geométrico. Leonard J. Schulman of CalTech, Casey Douglas of St. Mary's College, and Gary Lawlor of BYU all provided helpful insights during our research. Finally, we would like to thank our research advisor Frank Morgan, without whom this paper would never have been able to take shape.

2. Isoperimetric Problems in Manifolds with Density

A *density* on a Riemannian manifold is a nonnegative, lower semicontinuous function $\Psi(x)$ weighting both volume and hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the weighted volume and area are given by $dV = \Psi dV_0$, $dA = \Psi dA_0$. Manifolds with densities arise naturally in geometry as quotients of other Riemannian manifolds, in physics as spaces with different mediums, in probability as the famous Gauss space $\mathbb{R}^n$ with density $\Psi = ce^{-a^2r^2}$, and in a number of other places as well (see Morgan [M1, Ch. 18, M5]).

The generalized mean curvature of a manifold with density $\Psi(x) = e^{\psi(x)}$ is defined to be

$$H_\psi = H - \frac{1}{n-1}\frac{d\psi}{dn},$$

where H is the Riemannian mean curvature, as this corresponds to the first variation of weighted area [M5, Intro.], [M1, Ch. 18]. In two dimensions, the focus of this paper, this reduces to

$$\lambda = \kappa - \frac{d\psi}{dn},$$

where κ is the Riemannian curvature.

The *isoperimetric problem* on a manifold with density seeks to enclose a given weighted area with the least weighted perimeter. As in the Riemannian case, for a smooth density isoperimetric hypersurfaces have constant generalized curvature. The solution to the isoperimetric problem is known only for a few manifolds with density including Gauss space (see [M1, Ch. 18]) and the plane with a handful of different densities (see Cañete *et al.* [CMV, Sect. 3], Dahlberg *et al.* [DDNT, Thm 3.16], Engelstein *et al.* [EMMP, Cor. 4.9], Rosales *et al.* [RCBM, Thm. 5.2], and Maurmann and Morgan [MM, Cor. 2.2]).

3. Existence of Isoperimetric Regions in Manifolds with Density

In this section we extend some techniques used for proving existence of isoperimetric regions in $\mathbb{R}^n$ with density to certain noncomplete smooth manifolds. Our motivating example is a two dimensional cone with density r^p and the singular vertex removed. The main result is a general existence condition for minimizers in non-complete manifolds (Thm. 3.9). We use this to prove existence and regularity of minimizers in the cone with density r^p (Prop. 3.11). We also show existence and regularity of minimizers in the sector with disk density (Prop. 3.12).

We will be using the language of geometric measure theory but the results and the ideas of the important proofs should be accessible even to someone completely unfamiliar with the field. For such a reader we provide the following brief glossary:

- n-dimensional current $-$ a very general concept of a surface that allows some nice compactness properties.
- n-dimensional mass $-$ the equivalent of n-dimensional volume for an n-dimensional current. In our case it will always be equivalent to n-dimensional Hausdorff measure. We shall denote the mass of a current A by$\mathbf{M}(A)$.
- n-dimensional locally integral current $-$ an n-dimensional current with locally finite mass and locally finite boundary mass.

For a general reference for this section see [M1, Ch. 3-5,9].

There is a standard method for proving the existence of isoperimetric regions in $\mathbb{R}^n$ with some density (or possibly different densities for n-dimensional mass and $(n-1)$ dimensional mass). Namely, take a sequence of regions with a fixed weighted n-dimensional mass W_0 whose boundary masses approach the infimum weighted $(n-1)$-dimensional mass and then apply the compactness theorem of geometric measure theory to obtain a convergent subsequence. The trick of the

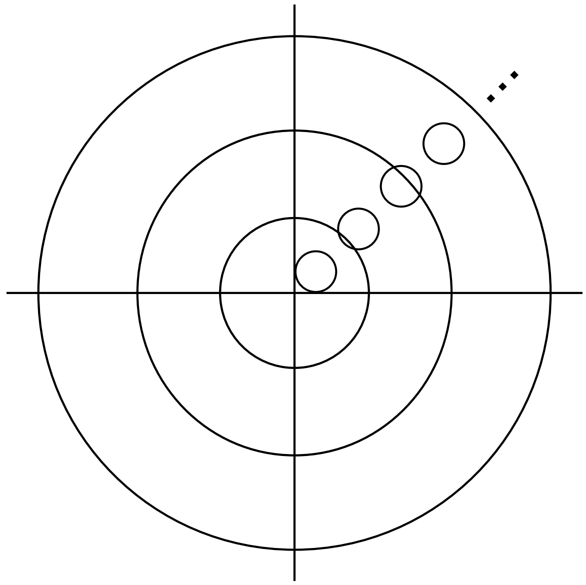


FIGURE 3.1. An example of mass escaping to infinity: the sequence of unit balls eventually lies outside any ball about the origin.

argument is then to show that the limit actually has n-dimensional weighted mass W_0. This is accomplished by showing that no weighted mass "escapes to infinity" in the limit, *i.e.*, eventually remains outside any ball. For an example of how mass could escape, consider $\mathbb{R}^2$ with unweighted area and perimeter. Then we obtain a sequence of regions containing area 1 and approaching the infimum perimeter by simply taking unit balls centered around (m, m), $m \in \mathbb{N}$ as in Figure 3.1. In this case all of the area escapes to infinity, and in fact this sequence converges to 0. Nevertheless, for many densities the argument can be carried through; see for example [RCBM, Sec. 2].

The essential idea of this technique is that weighted mass converges nicely inside compact sets and so it is only the behavior outside all compact sets that we have to worry about. The main obstacle in extending it to a non-complete Riemannian manifold is that now compactness fails in more places than just infinity. Specifically, at any point where it is not complete and the density blows up there is also the potential for weighted mass to disappear. In attempting to extend this proof technique our first steps are a technical build-up to Lemma 3.7 which provides the appropriate statement of this fact. It gives us two different places where weighted mass can escape: the regions where completeness fails, and infinity as before. We then make the trivial observation that weighted mass cannot disappear to any region of finite weighted mass. This leads to the main result of this section which is a general existence theorem with some straightforward hypotheses for isoperimetric regions where $(n-1)$-dimensional mass is unweighted but n-dimensional mass.

We will begin, however, on more solid ground with the non-existence of isoperimetric regions in the 2-dimensional cone over a circle with density r^p for $p \in [-2, 0)$. This case provides valuable insight and motivation towards proving existence in general. Our proof of non-existence is adapted with minor modifications from the planar proof in [CJQW, Prop. 4.2]. The first step is a change of coordinates.

Lemma 3.1. *A circular cone with density r^p is equivalent to a circular cone with Euclidean perimeter and area density r^q where $q \in (-\infty,-2) \cup (0,\infty)$ for $p \in [-2,0)\backslash\{-1\}$ and $q \in (-2,0)$ for $p \in (-\infty,-2) \cup (0,\infty)$.*

Proof. Making the change of coordinates $w = z^{p+1}/p+1$ gives Euclidean perimeter density and area density $cr^{-p/p+1}$. □

Theorem 3.2. *In the circular cone with density r^p for $p \in [-2,0)$ no isoperimetric regions exist.*

Proof. For $p \neq -1$ we make the change of coordinates from Lemma 3.1 so that we are in a cone with area density r^q, $q \in (-\infty,-2) \cup (0,\infty)$. Our method of showing nonexistence will be to fix a weighted area W_0 and then demonstrate that for any $\epsilon > 0$ there exists a region with perimeter less than or equal ϵ containing area W_0. To do this we consider two cases:

For $q \in (-\infty,-2]$, we note that any ball about the origin contains infinite weighted area. Thus we can take annuli around the origin with arbitrarily small perimeter bounding weighted area W_0.

For $q \in (0,\infty)$, take a ball anywhere in the cone of perimeter ϵ. If it contains weighted area greater than W_0 shrink it until it contains weighted area W_0. If it contains weighted area less than W_0 simply slide it out towards infinity until it has weighted area W_0, which will happen at some point since the ball of radius ϵ can have arbitrarily large weighted area if it is far enough from the origin.

For $p = -1$ we make the change of coordinates $w = \log z$ and remark that the same construction as in [CJQW, Prop. 4.2] for the plane works here. □

Remark. This same proof can be generalized to show that isoperimetric regions do not exist in higher dimensional spherical cones with standard $(n-1)$-dimensional mass and n-dimensional mass weighted by the density r^q for appropriate q.

There are two different reasons existence fails in this proof (thinking in the context of cones with Euclidean perimeter but some area density): in the one case there is infinite area around the origin, and in the other the density approaches infinity at infinity. For any $q \in (-2,0)$ there is finite area around the origin and the density approaches 0 at infinity. Carroll *et al.* [CJQW, Props. 4.3, 4.4] show existence in the plane for these cases and the natural conjecture is that existence also holds for any other circular cone. This basic observation that we want finite area about the origin and the area density to approach 0 at infinity will be useful to keep in mind as we formalize and generalize to higher dimensions.

Definition 3.3. Let M^n be a smooth n-dimensional manifold, not necessarily complete, and $Z \subset M^n$ a smooth compact n-dimensional submanifold with boundary. Then for any locally rectifiable current S on M^n define

$$\mathcal{F}_Z(S) = \inf\{\mathbf{M}(A)+\mathbf{M}(\partial B) | \operatorname{spt}(S-(A+\partial B)) \cap Z = \emptyset, A \in \mathfrak{R}_m M^n, B \in \mathfrak{R}_{m+1} M^n\}.$$

We give a basis for the local flat topology on $\mathbf{I}_m^{loc} M^n$ with the following open balls:

$$B_{Z,\delta}(T) = \{S \in \mathbf{I}_m^{loc} M^n | \mathcal{F}_Z(S-T) < \delta\}$$

where $T \in \mathbf{I}_m^{loc}M^n$, $\delta > 0$, and $Z \subset M^n$ is a smooth compact n-dimensional submanifold with boundary. In particular, a sequence $S_j \in \mathbf{I}_m^{loc}M^n$ converges to $L \in \mathbf{I}_m^{loc}M^n$ if and only if for every smooth compact submanifold with boundary $Z \subset M^n$ and every $\delta > 0$ there is a N such that for all $j > N$ we have $\mathcal{F}_Z(S_j - L) < \delta$.

Remark. This is the natural generalization of the local flat topology given by Morgan [M1, 9.1] for $\mathbb{R}^n$. It is important to switch from closed balls to smooth compact n-dimensional submanifolds because in a noncomplete manifold we can end up with a non-compact ball of radius R around a point − for instance a ball containing a deleted neighborhood of the vertex in the cone without its vertex.

Theorem 3.4 (Compactness Theorem)**.** *For a smooth n-dimensional Riemannian manifold M^n and an open cover U_i the set*

$$\{S \in \mathbf{I}_m^{loc}M^n | \mathbf{M}(S|_{U_i}) \leq c, \mathbf{M}(\partial S|_{U_i}) \leq c \text{ for all } i\}$$

is compact.

Remark. The Compactness Theorem is standard for integral currents in $\mathbb{R}^n$ (see [M1, 5.5]). Its extension to manifolds is discussed in a remark following the theorem and its extension to locally integral currents in $\mathbb{R}^n$ is discussed in [M1, 9.1].

For finding isoperimetric regions in manifolds with density we will want to consider usually not mass but rather weighted mass. That is, we take a positive lower semi-continuous density function f and then weighted mass $\mathbf{W}_f$ will be defined on a rectifiable set A by taking the sup where we integrate forms weighted by f. The next two lemmas establish some useful properties of weighted mass.

Lemma 3.5. *If $S_j \in \mathbf{I}_m^{loc}M^n$ is converging to L then $\mathbf{W}_f(L) \leq \liminf_{j\to\infty} \mathbf{W}_f(T_j)$ and $\mathbf{W}_f(\partial L) \leq \liminf_{j\to\infty} \mathbf{W}_f(\partial T_j)$.*

Proof. Follows from the definition of weighted mass that it is lower semicontinuous. □

Lemma 3.6. *If $S_j \in \mathbf{I}_n^{loc}M^n$ is converging to L then for any smooth n-dimensional compact submanifold with boundary $Z \subset M^n$ and lower semi-continuous positive density f we have $\lim_{j\to\infty} \mathbf{W}_f(S_j|_Z) = \mathbf{W}_f(L|_Z)$.*

Proof. In codimension 0 $\mathcal{F}_Z(S)$ simplifies to $\inf\{\mathbf{M}(A) | \mathrm{spt}(S - A)) \cap Z = \emptyset, A \in \mathfrak{R}_n\} = \mathbf{M}(S|_Z)$. So we see weighted mass is continuous on currents whose support is contained in Z. □

The next lemma formalizes the idea that weighted mass can only disappear outside compact sets.

Lemma 3.7. *Let M^n be a smooth n-dimensional manifold and let Z_k be an increasing sequence of smooth n-dimensional compact submanifolds with boundary such that $\bigcup_{k>0} Z_k = M^n$. Then if $S_j \in \mathbf{I}_n^{loc}M^n$ is converging to L and*

$$\lim_{k\to\infty} \lim_{j\to\infty} \mathbf{W}_f(S_j|_{(M^n \setminus Z_k)}) = 0,$$

then

$$\mathbf{W}_f(L) = \lim_{j\to\infty} \mathbf{W}_f(S_j).$$

Proof. Because the Z_k increase to M^n we have $\lim_{k\to\infty} \mathbf{W}_f(L|_{Z_k}) = \mathbf{W}_f(L)$. So, by Corollary 3.6 we get

$$\lim_{k\to\infty} \lim_{j\to\infty} \mathbf{W}_f(S_j|_{Z_k}) = \mathbf{W}_f(L)$$

which gives

$$\lim_{k\to\infty} \lim_{j\to\infty} (\mathbf{W}_f(S_j) - \mathbf{W}_f(S_j|_{(M^n \setminus Z_k)}) = \mathbf{W}_f(L)$$

and so splitting up the interior limit we obtain the desired result. □

The following establishes the simple observation that weighted mass cannot escape to any region of finite weighted mass.

Lemma 3.8. *Let U_k be a decreasing sequence of open subsets of M^n whose intersection is empty such that for some k_0 we have that $\mathbf{W}_f(U_{k_0})$ is finite. Then if $S_j \in \mathbf{I}_n^{loc} M^n$ is converging,*

$$\lim_{k\to\infty} \lim_{j\to\infty} \mathbf{W}_f(S_j|_{U_k}) = 0.$$

Proof. Note $\cup_{k>k_0} U_{k_0} \setminus U_k = U_{k_0}$ and so $\lim_{k\to\infty} \mathbf{W}_f(U_{k_0} \setminus U_k) = \mathbf{W}_f(U_{k_0})$ and thus $\lim_{k\to\infty} \mathbf{W}_f(U_k) = 0$. Since $\mathbf{W}_f(S_j|_{U_k})$ is bounded uniformly in j by $\mathbf{W}_f(U_k)$ we obtain the desired result. □

Theorem 3.9 will give us some conditions under which isoperimetric regions exist.

Theorem 3.9. *Suppose M^n is a smooth connected possibly non-complete n-dimensional Riemannian manifold with isoperimetric function I such that $\lim_{A\to\infty} I(A) = \infty$. Suppose furthermore that every closed geodesic ball of finite radius in M has finite volume and finite boundary area and there is some $C \in \mathbb{Z}^+$ and $x_0 \in M$ such that the complement of any closed geodesic ball of finite radius about x_0 contains finitely many connected components and at most C unbounded connected components. Then for standard boundary area and any lower semi-continuous positive volume density f such that*

(1) *for some $x_0 \in M^n$ $\sup\{f(x)|\text{dist}(x, x_0) > R\}$ goes to 0 as R goes to ∞;*
(2) *for some $\epsilon > 0, \{x|B(x, \epsilon)$ is not complete$\}$ has finite weighted volume;*

isoperimetric regions exist.

Proof. We first note that the second condition on the density and the condition that closed balls of finite radius have finite mass also implies that closed balls of finite radius have finite weighted mass since we can bound the weighted mass of a ball by the maximum density outside some neighborhood of the points where it fails to be complete times its mass plus the finite weighted mass contained in that neighborhood.

Now fix a weighted mass W_0 and let P_0 be the infimum $(n-1)$-dimensional boundary mass for W_0. Then consider a sequence S_j of locally integral currents such that $\mathbf{W}_f(S_j) = W_0$ and $\lim_{j\to\infty} \mathbf{M}(\partial S_j) = P_0$. We can apply Theorem 3.4 by noting that we can assume $\mathbf{M}(\partial S_j) < P_0 + 1$ and then taking any open cover with bounded n-dimensional mass. So, we will assume that S_j is converging to a limit L. By Lemma 3.5 we know that $\mathbf{M}(\partial L) \leq P_0$ and so it suffices to show that $\mathbf{W}_f(L) = W_0$. To this end we would like to apply Lemma 3.7. So, we must construct a suitable sequence of compact sets. To do this we will take an increasing sequence of closed balls about x_0, add any bounded connected components outside of it, and subtract off a decreasing sequence of open sets containing the areas where M^n fails to be complete.

We define $U_\epsilon = \text{int}\{x|B(x,\epsilon) \text{ is not complete}\}$ and note that $\cap_{\epsilon>0} U_\epsilon = \emptyset$ since M^n is locally diffeomorphic to $\mathbb{R}^n$. These are the sets we will subtract off. Next we we define V_t to be the union of $B(x_0, t)$ with every bounded connected component in the complement $M^n \backslash B(x_0, t)$. Since there are only finitely many bounded connected components outside of any such ball we obtain an increasing sequence of compact sets by taking $Z_k = V_k \backslash U_{1/k}$ for all $k \in \mathbb{Z}^+$. By Condition 3 and Lemma 3.8 no area can escape to $U_{1/k}$ and so it now suffices to show that no area can escape to infinity, *i.e.*, $M^n \backslash V_k$ as $k \to \infty$.

So let us consider what these sets $M^n \backslash V_k$ consist of with the goal of isolating "ends" of the manifolds in which we can work independently to show no weighted mass disappears. By definition each connected component is unbounded and by our original hypothesis there are at most C of these connected components. Furthermore, for $j < k$ each connected component of $M^n \backslash V_k$ is contained in some connected component of $M^n \backslash V_j$ and since every connected component of $M^n \backslash V_j$ is unbounded there is at least one connected component of $M^n \backslash V_k$ contained in each connected component of $M^n \backslash V_j$. But since there are at most C connected components of $M^n \backslash V_j$ for any j, we see that eventually these components stabilize in the following sense: there exists some k_0 such that $M^n \backslash V_{k_0}$ consists of m connected components $A_1, ..., A_m$ such that for any $k > k_0$ $A_i \backslash V_k$ has only one connected component for each $i \in \{1, ..., m\}$ and furthermore $M^n \backslash V_k$ consists exactly of the m connected components $A_1 \backslash V_k, ..., A_m \backslash V_k$. These A_i are the ends we will examine. It will suffice to consider each of them separately, and so from now on we work only with A_1.

If $\mathbf{W}_f(A_1) < \infty$ then we are done by Lemma 3.8. If not then our plan of attack will be to uniformly bound the weighted mass contained in $S_j|_{A_1}$ and then use the fact that the density goes to 0. Now, $\text{spt}(\partial(S_j|_{A_1})) \subset \text{spt}(\partial S_j) \cup \partial V_{k_0}$ and so since $\mathbf{M}(\partial V_{k_0})$ is finite and $\mathbf{M}(\partial S_j)$ is uniformly bounded we get $\mathbf{M}(\partial(S_j|_{A_1}))$ is uniformly bounded. So if we can show that $\mathbf{M}(S_j|_{A_1}) < \infty$ then we can apply the isoperimetric profile approaching infinity to get a uniform bound on $\mathbf{M}(S_j|_{A_1})$. The bound on $\mathbf{M}(\partial(S_j|_{A_1}))$ gives us that each connected component of $\text{spt}(\partial(S_j|_{A_1}))$ must be contained in some ball about x_0. The fact that A_1 minus this ball is connected means that everything outside of the ball is contained on one side of of this component and thus each component of $\text{spt}(\partial(S_j|_{A_1}))$ bounds one region of finite mass and one region of potentially infinite mass. Now if $\mathbf{M}(S_j|_{A_1})$ were infinity but each component of $S_j|A_1$ had finite mass we would reach a contradiction to our isoperimetric profile going to infinity because we could produce regions of arbitrarily large mass with boundary mass less than our bound on the boundary mass of $S_j|_{A_1}$. So, if $\mathbf{M}(S_j|_{A_1}) = \infty$ then $S_J|_{A_1}$ contains a component of infinite mass. But then its complement in A_1 has finite mass and thus will have finite weighted mass (here we need the fact that balls have finite weighted mass and the above established fact that the boundary of each component of $S_j|_{A_1}$ bounds one region with finite mass contained inside some ball and the other with possibly infinite mass). Since $\mathbf{W}_f(S_j|_{A_1})$ is also finite and $\mathbf{W}_f(A_1)$ is less than the sum of these two finite weighted masses we get a contradiction to our assumption that $\mathbf{W}_f(A_1)$ was infinite. Thus $\mathbf{M}(S_j|_{A_1}) < \infty$ and so we can use the fact that the isoperimetric profile is approaching infinity to get a uniform bound C on $\mathbf{M}(S_j|_{A_1})$

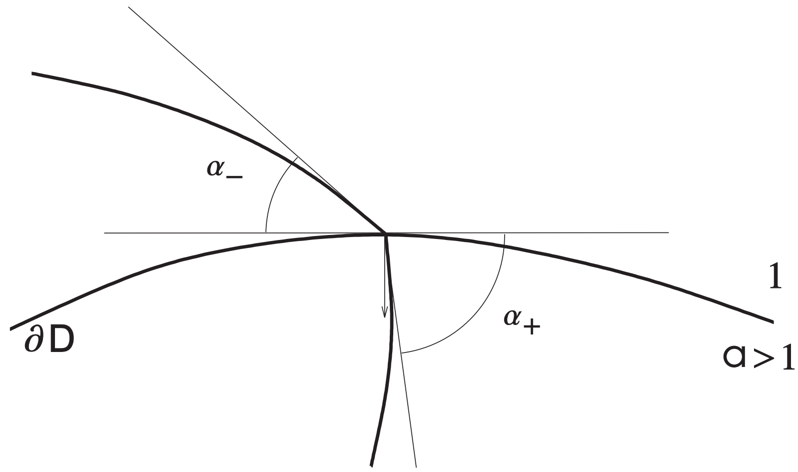


FIGURE 3.2. The Snell refraction rule for curves passing through the boundary of the unit disk. [CMV, Fig. 1] , used by permission, all rights reserved.

. Since

$$\begin{aligned} \mathbf{M}(S_j|_{A_1\backslash V_k}) &\leq \mathbf{M}(S_j|_{A_1\backslash V_k}) \cdot \sup\{f(x)|x \in A_1\backslash V_k\} \\ &\leq C \cdot \sup\{f(x)|x \in A_1\backslash V_k\}, \end{aligned}$$

and the sup goes to 0 as k goes to infinity we are done. □

Remark. The condition on balls where completeness fails in Theorem 3.9 can be softened slightly: if we consider the metric completion of M^n, $\bar{M}^n$, and let $K = \bar{M}^n \backslash M^n$, then all that is required is a sequence of open neighborhoods decreasing to K with finite weighted mass.

We state now a regularity result by Morgan [M6, Cor. 3.8, Sect. 3.10].

Theorem 3.10. *For $n \leq 6$, let S be an n-dimensional isoperimetric hypersurface in a manifold M with smooth Riemannian metric and smooth positive area and perimeter densities. Then S is a smooth submanifold.*

The next proposition gives the existence and regularity of minimizers in the main case that we examine in the rest of the paper.

Proposition 3.11. *In the circular cone with density r^p for $p \in (-\infty, -2) \cup (0, \infty)$ isoperimetric curves exist and are smooth.*

Proof. By Lemma 3.1 this is equivalent to a circular cone with Euclidean perimeter and area density r^q for $q \in (-2, 0)$. Furthermore it suffices to consider the cone with deleted vertex. Then it is easy to see that we meet the criteria of Theorem 3.9. Smoothness follows from Theorem 3.10. □

We finish with the existence and regularity of minimizers for a certain piecewise constant density examined in Section 7.

Proposition 3.12. *In the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, isoperimetric curves exist for any given area. These isoperimetric* *curves are smooth except at the boundary of D, where they obey the following Snell refraction rule (see Fig. 3.2):*

$$\frac{\cos \alpha_+}{\cos \alpha_-} = \frac{1}{a},$$

where a_+ is the angle of intersection from inside of D and α_- is the angle of intersection from outside of D.

Proof. Cañete *et al.* [CMV, Thm. 3.18] prove existence for the plane. The same result and proof hold for the θ_0-sector. The regularity and Snell refraction rule follow from [CMV, Prop. 2.14]. □

4. Isoperimetric Regions in Sectors with Density

We study properties of isoperimetric regions in planar sectors with radial densities. Proposition 4.2 shows there is a one-to-one correspondence between minimizers in the θ_0-sector and in the $2\theta_0$-cone, modulo rotations. Propositions 4.4 and 4.5 provide some regularity results.

Lemma 4.1. *Given an isoperimetric region in the $2\theta_0$-cone with density $f(r)$, there exist two rays from the origin separated by an angle of θ_0 that divide both the area and perimeter of the region in half.*

Proof. First we show that there are two such rays that separate the area of the region in half. Take any two rays from the origin separated by an angle of θ_0. Rotate them around, keeping an angle of θ_0 between them. The area between them varies continuously as we rotate the rays, and thus so does their difference, A. By the time we rotate the rays by an angle of θ_0, A has changed to $-A$. By the intermediate value theorem, at some point the difference must be 0, implying there are two rays separated by an angle θ_0 that divide the area in half. If one side had less perimeter than the other, we could reflect it to obtain a region with the same area and less perimeter than our original region, violating the condition that it was isoperimetric. □

Proposition 4.2. *An isoperimetric region of area $2A$ in the cone of angle $2\theta_0$ with density $f(r)$ has perimeter equal to the twice the perimeter of an isoperimetric region of area A in the sector of angle θ_0 with density $f(r)$. Indeed, the operation of doubling a sector to form a cone provides a one-to-one correspondence between isoperimetric regions in the sector and isoperimetric regions in the cone, modulo rotations.*

Proof. Given an isoperimetric region in the sector, take its reflection into the cone to obtain a region in the cone with twice the area and twice the perimeter. This region must be isoperimetric for the cone, for if there were a region in the cone with less perimeter for the same area we could divide its area in half by two rays separated by an angle θ_0 as in Lemma 4.1, take the side with at most half the perimeter to obtain a region in the θ_0-sector with the same area and less perimeter than our original isoperimetric region.

Conversely, given an isoperimetric region in the cone, divide its area and perimeter in half by the two rays described in Lemma 4.1. Both regions must be isoperimetric in the sector, for if there were a region with less perimeter for the same area, taking its double would yield a region in the cone with less perimeter than our original region for the same area. □

We shall find many occasions to use the following proposition of Dahlberg *et al.*

Proposition 4.3. [DDNT, Lem. 2.1] *Consider* $\mathbb{R}^2 - \{0\}$ *with smooth radial density* $e^{\psi(r)}$. *A constant-generalized-curvature curve is symmetric under reflection across every line through the origin and a critical point of* r.

Proof. This holds from the uniqueness of ordinary differential equations. We note that the same proof holds in the sector with radial density. □

Proposition 4.4. *In the* θ_0*-sector with smooth density* $f(r)$*, isoperimetric curves meet the boundary perpendicularly.*

Proof. By Proposition 4.2 there is a one-to-one correspondence between minimizers in the sector and the cone, meaning the double of this curve in the cone of angle $2\theta_0$ is minimizing and hence smooth (Prop. 3.11). Therefore, the original curve meets the boundary perpendicularly. □

Proposition 4.5. *In the* θ_0*-sector with smooth density* $f(r)$*, if isoperimetric curves are nonconstant polar graphs, they do not contain a critical point on the interior.*

Proof. Assume there is an isoperimetric curve r with a critical point on the interior. By Proposition 4.4, $\dot{r}(0) = \dot{r}(\theta_0) = 0$. Since constant generalized curvature curves are symmetric under reflection across a line through the origin and a critical point of r (Prop. 4.3), in the cone of angle $2\theta_0$ this curve has at least four critical points. By symmetry, critical points must be strict extrema. Let C be a circle about the axis intersecting the curve in at least four points. C divides the curve into at least two regions above C and two regions below C. Interchanging one region above C with a region below C results in a region with the same perimeter and area whose boundary is not smooth. Since isoperimetric curves must be smooth, r cannot be a minimizer. □

Corollary 4.6. *If an isoperimetric curve* $r(\theta)$ *in the* θ_0*-sector with density is a nonconstant polar graph,* r *must be strictly monotonic.*

Remark. Proposition 4.5 and Corollary 4.6 strengthen some arguments of Adams *et al.* [ACDLV, Lem. 3.6].

5. The Isoperimetric Problem in Sectors with Density r^p

Our main theorem, Theorem 5.19, characterizes isoperimetric regions in a planar sector with density r^p, $p > 0$. The subsequent results consider an analytic formulation of the problem.

Proposition 5.1. *In the half plane with density* r^p, $p > 0$*, semicircles through the origin are the unique isoperimetric curves.*

Proof. By Proposition 4.2, there is a one-to-one correspondence between minimizers in the θ_0-sector and the $2\theta_0$-cone; in particular there is a correspondence between minimizers in the half plane and minimizers in the 2π-cone, *i.e.*, the plane. Since circles through the origin are the unique minimizers in the plane (see Sect. 1.1), semicircles through the origin are the unique minimizers in the half plane. □

Proposition 5.2. *For density* r^p*, if the circle about the origin is not uniquely isoperimetric in the* θ_0*-sector, for all* $\theta > \theta_0$ *it is not isoperimetric.*

Proof. Let r be a non-circular isoperimetric curve in the θ_0-sector, and let C be a circle bounding the same area as r. For any angle $\theta > \theta_0$, transition to the θ-sector via the map $\alpha \to \alpha\theta/\theta_0$. This map multiplies area by θ/θ_0, and scales tangential perimeter. Therefore, if r had the same or less perimeter than C in the θ_0-sector, its image under this map has less perimeter than a circle about the origin in the θ-sector. □

Proposition 5.3. *In the θ_0-sector with density r^p, $p > 0$, an isoperimetric region contains the origin, and its boundary is a polar graph.*

Proof. Work in Euclidean coordinates via the mapping $w = z^{p+1}/(p+1)$. Here perimeter is Euclidean perimeter and the area density is cw^{-q} where $q = p/(p+1)$. Since the area density is strictly decreasing away from the origin, an isoperimetric region must contain the origin. Any minimizer is bounded by a smooth curve of constant generalized curvature (Thm. 3.10), and since generalized curvature is just Riemannian curvature divided by the area density [CJQW, Def. 3.1], the Riemannian curvature does not change sign, and the curve is convex. Thus an isoperimetric curve is a polar graph in Euclidean coordinates and hence in the original coordinates. Note that all constant generalized curvature curves are convex in Euclidean coordinates. □

Definition 5.4. An *unduloid* is a nonconstant positive polar graph with constant generalized curvature.

Here we state another result from Dahlberg *et al.* that will be useful in classifying the potential minimizers for the θ_0-sector with density r^p.

Proposition 5.5. [DDNT, Prop. 2.11] *In a planar domain with density r^p, $p > 0$, if a constant generalized curvature closed curve passes through the origin, it must be a circle.*

Lemma 5.6. *In the θ_0-sector with density r^p, isoperimetric curves are either circles about the origin, semicircles through the origin, or unduloids.*

Proof. By Proposition 5.3, minimizers must be polar graphs with constant generalized curvature bounding regions that contain the origin. Therefore it can either be constant, a circle, or nonconstant, an unduloid. If the curve goes through the origin, it must be part of a circle through the origin (Prop. 5.5). However, to meet regularity conditions, the curve must be an integer number of semicircles. Since one semicircle is better than n semicircles, the minimizer will be a single semicircle through the origin. □

Proposition 5.7. [DDNT, Lem. 3.7] *In the plane with density r^p, $p > 0$, the least-perimeter 'isoperimetric' function $I(A)$ satisfies*

$$I(A) = cA^{\frac{p+1}{p+2}}.$$

Remark. While this result is stated in the plane, it also holds in the sector.

Theorem 5.8. *In the θ_0-sector with density r^p, $p > 0$, circular arcs are isoperimetric for $\theta_0 = \pi/(p+1)$.*

Proof. Transition to Euclidean coordinates, where θ_0-sector gets mapped to the half plane. Assume some $r(\theta)$ other than the circle is isoperimetric. By Proposition 4.5, $r(0) \neq r(\pi)$, and $\dot{r}(\theta) = 0$ at 0 and π, and nowhere else. Reflect r over the x-axis, obtaining a closed curve. By the four-vertex theorem, this curve has at least four extrema of classical curvature. Generalized curvature in Euclidean coordinates is Riemannian curvature divided by the area density [CJQW, Def. 3.1]. That is:

$$\kappa_\varphi = cr^{p/p+1}\kappa$$

for some $c > 0$. At an extremum of Riemannian curvature we see

$$0 = \frac{d}{d\theta}\kappa_\varphi = \kappa' cr^{p/p+1} + c' r^{-1/p+1}\dot{r}\kappa = c' r^{-1/p+1}\dot{r}\kappa,$$

which implies either $\dot{r} = 0$ or $\kappa = 0$. However, if $\kappa = 0$, the curve is the geodesic, which is a straight line in Euclidean coordinates, which cannot be isoperimetric. Therefore $\dot{r} = 0$, meaning r must have a critical point other than 0 and π, so it cannot be isoperimetric. □

Remark. After finding this proof and examining the isoperimetric inequality in Proposition 5.27, we came across a more geometric proof. We show that in the Euclidean θ_0-sector with area density $cr^{-p/p+1}$, circles about the origin are isoperimetric for $\theta_0 = \pi$. When $p = 0$, a semicircle about the origin is a minimizer. Now, for any $p > 0$, suppose some region R is a minimizer. Take a semicircle about the origin bounding the same Euclidean area; clearly it will have less perimeter. However, it also has more weighted area, because we have moved sections of R that were further away from the origin towards the origin. Since the area density is strictly decreasing in r, we must have increased area. Therefore the circle about the origin is the minimizer for the Euclidean θ_0-sector with area density $cr^{-p/p+1}$ for $\theta_0 = \pi$, implying circular arcs are the minimizers in the $\pi/(p+1)$-sector with density r^p.

Proposition 5.9. *When $p = 1$ circles about the origin are minimizing in the sector of 2 radians with density r^p.*

Proof. Morgan [M3, Prop. 1] proves that in cones over the square torus $\mathbb{T}^2 = \mathbb{S}^1(a) \times \mathbb{S}^1(a)$ balls about the origin are minimizing as long as $|\mathbb{T}^2| \leq |\mathbb{S}^2(1)|$. We note that this proof still holds for rectangular tori $\mathbb{T}^2 = \mathbb{S}^1(a) \times \mathbb{S}^1(b)$, $a \geq b$ so long as the ratio $a : b$ is at most $4 : \pi$. Taking the cone over the rectangular torus with side lengths 4 and π, and modding out by the shorter copy of $\mathbb{S}^1$ we get the cone over an angle 4 with density πr. Since balls about the origin are minimizers in the original space, this implies that their images, circles about the origin, are minimizing in this quotient space. □

Remark. Morgan and Ritoré [MR, Rmk. 3.10] ask whether $|M^n| \leq |\mathbb{S}^n(1)|$ is enough to imply that balls about the origin are isoperimetric in the cone over M. Trying to take the converse to the above argument we found an easy counterexample to this question. Namely taking M to be a rectangular torus of area 4π with one very long direction and one very short direction, we see that balls about the origin are not minimizing, as you can do better with a circle through the origin cross the short direction.

Along the same lines of Proposition 5.9 one might hope to get bounds for other values of p by examining when balls about the origin are minimizing in cones over $\mathbb{S}^1(\theta) \times M^p$ for M compact with a transitive isometry group and then modding out by the symmetry group of M to get the cone over $\mathbb{S}^1(\theta)$ with density proportional to r^p. In order to see when balls about the origin are minimizing in the cone over $\mathbb{T}^2$, Morgan uses the Ros product theorem with density [M2, Thm. 3.2], which requires the knowledge of the isoperimetric profile of the link (in his case $\mathbb{T}^2$). One such manifold of the form $\mathbb{S}^1 \times M$ for which the isoperimetric problem is solved is $\mathbb{S}^1 \times \mathbb{S}^2$ [PR, Thm. 4.3]. The three types of minimizers in $\mathbb{S}^1 \times \mathbb{S}^2$ are balls or complements of balls, tubular neighborhoods of $\mathbb{S}^1 \times \{point\}$, or regions bounded by two totally geodesic copies of $\mathbb{S}^2$. By far the most difficult of the three cases to deal with are the balls, where unlike the other two cases we cannot explicitly compute the volume and surface area. Fixing the volume of $\mathbb{S}^1 \times \mathbb{S}^2$ as $2\pi^2 = |\mathbb{S}^3|$ we can apply the same argument Morgan uses in [M3, Lemma 2] to show that if the sectional curvature of $\mathbb{S}^1 \times \mathbb{S}^2$ is bounded above by 1 (by taking $\mathbb{S}^2$ large and $\mathbb{S}^1$ small), then balls in $\mathbb{S}^1 \times \mathbb{S}^2$ do worse than balls in $\mathbb{S}^3$, as desired, but unfortunately tubular neighborhoods of $\mathbb{S}^1$ then sometimes beat balls in $\mathbb{S}^3$ for certain volumes, making it so we cannot apply the Ros product theorem. So without a better way to deal with balls in $\mathbb{S}^1 \times \mathbb{S}^2$ without such a strong assumption on the sectional curvature, this method does not work even for the $p = 2$ case. Perhaps there is some other way to prove when balls about the vertex are isoperimetric in the cone over $\mathbb{S}^1 \times \mathbb{S}^2$ without the Ros product theorem.

Proposition 5.10. *In the θ_0-sector with density r^p, semicircles through the origin are not isoperimetric for $\theta_0 < (\pi/2)\,(p+2)\,/\,(p+1)$.*

Proof. In Euclidean coordinates, semicircles through the origin terminate at the angle $\theta = (\pi/2)\,(p+1)$. Since the semicircle approaches this axis tangentially, for any $\theta_0 < (\pi/2)\,(p+2)/(p+1)$, there is a line normal to the boundary $\theta_0(p+1)$ in Euclidean coordinates which intersects the semicircle at a single point b. Replacing the segment of the semicircle from b to the origin with this line increases area while decreasing perimeter. Therefore semicircles are not minimizing. □

Lemma 5.11. *If the θ_0-sector with density r^p has isoperimetric ratio I_0 and*

$$I_0 < (p+2)^{p+1}\theta_0 \left[\left(\frac{1}{p+2}\right) + \left(\frac{p+1}{p+2}\right)\theta_0\right]^{p+2},$$

then there exists an $\epsilon > 0$ such that the isoperimetric ratio of any $(\theta_0 + t)$-sector with $0 \le t < \epsilon$ is greater than or equal to I_0.

Proof. Consider an isoperimetric curve γ in the (θ_0+t)-sector bounding area 1. By reflection we can assume it is nonincreasing (Cor. 4.6). We partition the (θ_0+t)-sector into a θ_0-sector followed by a t-sector and we let α_0 denote the area bounded by γ in the θ_0-sector and α_t the area bounded by γ in the $t-$sector. Note $\alpha_0+\alpha_t = 1$. Then we can bound the isoperimetric ratio I_t of the (θ_0+t)-sector by using the isoperimetric ratios I_0 for the θ_0-sector and R_t for the t-sector as follows:

$$I_t^{1/(p+2)} = P(\gamma) \ge (I_0\alpha_0^{p+1})^{1/(p+2)} + (R_t\alpha_t^{p+1})^{1/(p+2)}.$$

Since the radius is nonincreasing we know that the θ_0-sector contains at least its angular proportion of the area and thus $\alpha_0 \ge \theta_0/(\theta_0+t)$. We now substitute $\alpha_t = 1 - \alpha_0$ and look at the right side of the inequality as a function of α_0:

$$f_t(\alpha_0) = (I_0\alpha_0^{p+1})^{1/(p+2)} + (R_t(1-\alpha_0)^{p+1})^{1/(p+2)}.$$

This function is concave, so it attains its minimum at an endpoint. Since α_0 is bounded between $\theta_0/(\theta_0+t)$ and 1, we see that $I_t^{1/(p+2)}$ is greater than or equal to the minimum of $f_t(\theta_0/(\theta_0+t))$ and $f(1)$. Since $f(1) = I_0^{1/(p+2)}$, we want to show that there is some ϵ such that $t < \epsilon$ implies $f_t(\theta_0/(\theta_0+t)) \geq f(1) = I_0^{1/(p+2)}$. So we define a new function

$$g(t) = f_t(\theta_0/(\theta_0+t)) - I_0^{1/(p+2)}.$$

We want to show that there is some positive neighborhood of 0 where $g(t) > 0$. For $t < \pi/(p+1)$, isoperimetric regions are circular arcs, so for $t \in (0, \delta)$, $R_t = (p+2)^{p+1}t$ and g is differentiable. Furthermore, $\lim_{t\to 0} g(t) = 0$ and so it suffices to prove that $\lim_{t\to 0} g'(t) > 0$. We calculate

$$\begin{aligned}\lim_{t\to 0} g'(t) \quad &= \quad -I_0^{1/(p+2)}\left(\frac{p+1}{p+2}\right)\theta_0^{-1} + \\ &\quad (p+2)^{(p+1)/(p+2)}\left[\left(\frac{1}{p+2}\right)\theta_0^{-(p+1)/(p+2)} + \left(\frac{p+1}{p+2}\right)\theta_0^{1/(p+2)}\right]\end{aligned}$$

and deduce that this is greater than 0 if and only if

$$I_0 < (p+2)^{p+1}\theta_0\left[\left(\frac{1}{p+2}\right) + \left(\frac{p+1}{p+2}\right)\theta_0\right]^{p+2}.$$

□

Corollary 5.12. *If the θ_0-sector with density r^p has isoperimetric ratio I_0 and*

$$I_0 < (p+2)^{p+1}\theta_0\left[\left(\frac{1}{p+2}\right) + \left(\frac{p+1}{p+2}\right)\theta_0\right]^{p+2},$$

then there exists an $\epsilon > 0$ such that the isoperimetric ratio is nondecreasing on the interval $[\theta_0, \theta_0+\epsilon)$.

Proof. The result follows from the continuity of the isoperimetric ratio in θ_0 and Lemma 5.11. □

Proposition 5.13. *For fixed $p > 0$, the isoperimetric ratio of the θ_0-sectors with density r^p is a nondecreasing function of θ_0 for $\theta_0 > 1$.*

Proof. When $\theta_0 > 1$ we have

$$(p+2)^{p+1}\theta_0 < (p+2)^{p+1}\theta_0\left[\left(\frac{1}{p+2}\right) + \left(\frac{p+1}{p+2}\right)\theta_0\right]^{p+2}.$$

Since $(p+2)^{p+1}\theta_0$ is the isoperimetric ratio of the circular arc in the θ_0-sector we see that for these θ_0 the isoperimetric ratios I_0 always satisfy the conditions of Corollary 5.12. So, for any $\theta_1 > \theta_0$ we can apply Corollary 5.12 to get some nondecreasing interval starting at θ_1 and then extend it on the right to a maximal nondecreasing interval. Since the isoperimetric ratio is continuous in θ_0 we can take this interval to be closed and then if it is bounded we get a contradiction by applying Corollary 5.12 to its right endpoint. So, we see that for each $\theta_1 > \theta_0$ the isoperimetric ratio is nondecreasing on $[\theta_1, \infty)$ and thus we deduce that the isoperimetric ratio is nondecreasing on (θ_0, ∞). □

Corollary 5.14. *In the θ_0-sector with density r^p, if the semicircle through the origin is isoperimetric, it uniquely minimizes for all $\theta > \theta_0$.*

Proof. By Proposition 5.13, the isoperimetric ratio is nondecreasing for $\theta_0 > 1$. However, for curves that don't terminate, Lemma 5.11 and Proposition 5.13 actually show the isoperimetric ratio is strictly increasing for $\theta_0 > 1$. Since the semicircle through the origin does not exist before $\pi/2 > 1$, if it is a minimizer for θ_0, it minimizes uniquely for all angles greater than θ_0. □

Corollary 5.15. *The semicircle is the unique minimizer in the "half-infinite parking garage"* $\{(\theta,\ r) | \theta \geq 0,\ r > 0\}$ *with density* r^p.

Proof. Suppose γ is a minimizer in the half-infinite parking garage. Then for any $\theta_0 \geq \pi$ the restriction of γ to the θ_0 sector has isoperimetric ratio greater than that of the semicircle. Since $\lim_{\theta_0 \to \infty} P(\gamma|_{\theta_0}) = P(\gamma)$ and $\lim_{\theta_0 \to \infty} A(\gamma|_{\theta_0}) = A(\gamma)$ the limit of the isoperimetric ratios of $\gamma|_{\theta_0}$ is the isoperimetric ratio of γ and so we see it is also greater than that of the semicircle. Since the semicircle exists in the half-infinite parking garage we are done. □

Remark. Studying the isoperimetric ratio turns out to be an extremely useful tool in determining the behavior of semicircles for $\theta_0 > \pi$. However, it is not the only such tool. Here we give an entirely different proof that semicircles minimize for all $n\pi$-sectors.

Proposition 5.16. *In the* θ_0*-sector with density* r^p, $p > 0$, *even when allowing multiplicity greater than one, isoperimetric regions will not have multiplicity greater than one.*

Proof. A region R with multiplicity may be decomposed as a sum of nested regions R_j with perimeter and area [M1, Fig. 10.1.1]:

$$P(R) = \sum P(R_j),$$

$$A(R) = \sum A(R_j).$$

Let R' be an isoperimetric region of multiplicity one and the same area as R. By scaling, for each region $P_j \geq cA_j^{(p+1)/(p+2)}$, where $c = P_c^{p+2}/A_R^{p+1}$ (Prop. 5.7). By concavity

$$P(R) = \sum P(R_j) \geq c \sum \left(A(R_j)^{\frac{p+1}{p+2}} \right) \geq c \left(\sum A(R_j) \right)^{\frac{p+1}{p+2}} = P(R'),$$

with equality only if R has multiplicity one. Therefore no isoperimetric region can have multiplicity greater than one. □

Remark. For $p < -2$ the isoperimetric function $I(A) = cA^{(p+1)/(p+2)}$ is now convex. Therefore, regions with multiplicity greater than one can do arbitrarily better than regions with multiplicity one.

Corollary 5.17. *In the* $n\pi$*-sector* ($n \in \mathbb{Z}$), *semicircles through the origin are uniquely isoperimetric.*

Proof. Assume there is an isoperimetric curve $r(\theta)$ bounding a region R which is not the semicircle. Consider R as a region with multiplicity in the half plane by taking $r(\theta) \to r(\theta \bmod \pi)$. Since semicircles are the minimizers in the half plane, by Proposition 5.16, r cannot be isoperimetric. This implies that r could not have been isoperimetric in the $n\pi$-sector. □

Proposition 5.18. *In the θ_0-sector with density r^p, $p > -1$, circles about the origin have nonnegative second variation if and only if $\theta_0 \leq \pi/\sqrt{p+1}$. When $p \leq -1$, circles about the origin always have nonnegative second variation.*

Proof. By Proposition 4.2 we think of the θ_0-sector as the cone of angle $2\theta_0$. A circle of radius r in the θ_0-sector corresponds with a circle about the axis with radius $r\theta_0/\pi$, giving the cone the metric $ds^2 = dr^2 + (r\theta_0/\pi)^2 d\theta^2$. For a smooth Riemannian disk of revolution with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$ and density $e^{\psi(r)}$, circles of revolution at distance r have nonnegative second variation if and only if $Q(r) = f'(r)^2 - f(r)f''(r) - f(r)^2\psi''(r) \leq 1$ [EMMP, Thm. 6.3]. This corresponds to $(\theta_0/\pi)^2 + p(\theta_0/\pi)^2 \leq 1$, which, for $p \leq -1$, always holds. When $p > -1$, the condition becomes $\theta_0 \leq \pi/\sqrt{p+1}$, as desired. □

The following theorem is the main result of this paper.

Theorem 5.19. *Given $p > 0$, there exist $0 < \theta_1 < \theta_2 < \infty$ such that in the θ_0-sector with density r^p, isoperimetric curves are (see Fig. 1.1):*

1. *for $0 < \theta_0 < \theta_1$, circular arcs about the origin,*
2. *for $\theta_1 < \theta_0 < \theta_2$, unduloids,*
3. *for $\theta_2 < \theta_0 < \infty$, semicircles through the origin.*

Moreover,

$$\pi/(p+1) < \theta_1 \leq \pi/\sqrt{p+1},$$
$$\pi(p+2)/(2p+2) \leq \theta_2 \leq \pi.$$

When $p = 1$, $\theta_1 \geq 2 > \pi/2 \approx 1.57$.

Proof. By Lemma 5.6, minimizers must be circles, unduloids, or semicircles. As θ increases, if the circle is not minimizing, it remains not minimizing (Prop. 5.2). If the semicircle is minimizing, it remains uniquely minimizing (Cor. 5.14). Therefore transitional angles $0 \leq \theta_1 \leq \theta_2 \leq \infty$ exist. Strict inequalities are trivial consequences of the following inequalities.

To prove $\theta_1 > \pi/(p+1)$, note that otherwise, because circular arcs are the unique minimizers for $\theta_0 = \pi/(p+1)$ (Thm. 5.8), there would have to be a family of other minimizers (unduloids, because certainly not semicircles) approaching the circle. This family approaches smoothly because by the theory of differential equations constant generalized curvature curves depend smoothly on their parameters. This would imply that the circle has nonpositive second variation, contradicting Proposition 5.18.

To prove $\theta_1 \leq \pi/\sqrt{p+1}$, recall that circular arcs do not have nonnegative second variation for $\theta_0 > \pi/\sqrt{p+1}$ (Prop. 5.18). To prove $\theta_2 \geq \pi(p+2)/(2p+2)$, just recall that semicircles cannot minimize for $\theta_0 < \pi(p+2)/(2p+2)$ (Prop. 5.10). To prove $\theta_2 \leq \pi$, recall that semicircles minimize for $\theta_0 = \pi$ (Prop. 5.1). Finally, when $p = 1$, circles minimize for $\theta_0 = 2$ (Prop. 5.9). □

Remark. For $p < -2$, circles about the origin minimize for all sectors. The proof in Euclidean coordinates given by Carroll *et al.* [CJQW, Prop. 4.3] generalizes immediately from the plane to the sector. This proof can also be translated into the original coordinates.

We conjecture that the circle is minimizing as long as it has nonnegative second variation, and that the semicircle is minimizing for all angles greater than $\pi\,(p+2)\,/\,(2p+2)$.

Conjecture 5.20. *In Theorem 5.19, the transitional angles θ_1, θ_2 are given by $\theta_1 = \pi/\sqrt{p+1}$ and $\theta_2 = \pi(p+2)/(2p+2)$.*

Remark. This conjecture is supported by numeric evidence as in Figure 1.2. Our Maple program (Sect. 10) predicts the circular arc stops minimizing at exactly $\pi/\sqrt{p+1}$, and predicts the semicircle begins to minimize very close to $\pi(p+2)/(2p+2)$.

Remark. Our minimizing unduloids give explicit examples of the abstract existence result of Rosales *et al.* [RCBM, Cor. 3.13] of isoperimetric regions not bounded by lines or circular arcs.

One potential avenue for proving this conjecture is discussed in 6.14. We also believe the transition between the circle and the semicircle is parametrized smoothly by curvature, which is discussed in the remark following Proposition 6.13.

Corollary 5.22 gives an example of the applicability of Theorem 5.19, as suggested to us by Antonio Cañete, extending to certain polygons with density an isoperimetric theorem for polytopes given by Morgan [M2, Thm. 3.8]. First we need a small lemma:

Lemma 5.21. *Consider a polygon in the Euclidean plane. Then there exists a $c > 0$ such that any region in the polygon bounding area A less than half the area of the polygon with perimeter P satisfies the following inequality:*

$$\frac{P^2}{A} \geq c.$$

Proof. This inequality is known for the circle. Mapping the polygon to a circle such that the factor by which distance is stretched is bounded above and below, we see it holds in a polygon, although the constant c depends on the polygon. □

Corollary 5.22. *Consider a polygon with a vertex of angle θ_0 located at the origin in the plane with density r^p, $p > 0$. For sufficiently small area, the isoperimetric curve bounding that area will be the same as in the θ_0-sector with density r^p.*

Proof. Let r_0 be small enough so that $B(0, r_0)$ intersects only one vertex (the origin) and two edges. Consider areas small enough that any region of that area has less than half the Euclidean area of the polygon, and so that an isoperimetric region of that area has perimeter less than $(r_0/2)^{p+1}$, so that any curve from the circle of radius $r_0/2$ to the circle of radius r_0 has more weighted perimeter. An isoperimetric region inside the circle of radius r_0 satisfies

$$P = cA^{(p+1)/(p+2)}$$

(Prop. 5.7). We claim that any region outside the circle of radius $r_0/2$ satisfies $P \geq c' A^{1/2}$.

Let R be a (possibly disconnected) isoperimetric region in the polygon with perimeter P and area A. We note that every component of R is either inside $B(0, r_0)$ or outside $B(0, r_0/2)$. Suppose there is a component outside $B(0, r_0/2)$ with area A_1 and perimeter P_1. By Lemma 5.21 and because r is bounded above and below this component satisfies the following inequality:

$$P_1 \geq c'' A_1^{1/2}.$$

Since for sufficiently small A_1

$$cA_1^{(p+1)/(p+2)} < c'' A_1^{1/2},$$

we see the best curve inside $B(0,r)$ bounding area A_1 will have less perimeter than the best curve outside $B(0,r/2)$ bounding area A_1. This implies R could have the same area with less perimeter as a region with multiplicity in $B(0,r)$. This is the same as a region with multiplicity in the sector, which by Proposition 5.16 cannot be isoperimetric. Therefore for sufficiently small area an isoperimetric curve must lie inside $B(0,r)$, meaning the isoperimetric curve will be the same as that in the sector. □

Proposition 5.23. *For any $n \in \mathbb{R}\backslash\{0\}$ the θ_0-sector with perimeter density r^p and area density r^q is equivalent to the $|n|\theta_0$-sector with perimeter density $r^{[(p+1)/n]-1}$ and area density $r^{[(q+2)/n]-2}$.*

Proof. Make the coordinate change $w = z^n/n$. □

Using Proposition 5.23 and our results in the sector (Thm. 5.19) we obtain the following proposition:

Proposition 5.24. *In the plane with perimeter density r^k, $k > -1$, and area density r^m the following are isoperimetric curves:*

(1) *for $m \in (-\infty, -2] \cup (2k, \infty)$ there are none.*
(2) *for $k \in (-1, 0)$ and $m \in (-2, 2k)$ the circle about the origin.*
(3) *for $k \in [0, \infty)$ and $m \in (-2, k-1]$ the circle about the origin.*
(4) *for $k \in [0, \infty)$ and $m \in [k, 2k]$ pinched circles through the origin.*

We also obtain a conjecture on the undecided area density range between $k-1$ and k based on our conjecture on sectors (Conj. 5.20).

Conjecture 5.25. *In the plane with perimeter density r^k, $k > -1$, and area density r^m the following are isoperimetric curves:*

(1) *for $k \in [0, \infty)$ and $m \in (k-1, k-1+\frac{1}{k+1}]$ the circle about the origin.*
(2) *for $k \in [0, \infty)$ and $m \in (k-1+\frac{1}{k+1}, k-1+\frac{k+1}{2k+1})$ unduloids.*
(3) *for $k \in [0, \infty)$ and $m \in [k-1+\frac{k+1}{2k+1}, k]$ pinched circles through the origin.*

Remark. Along the lines of Conjecture 5.25, the circular arc being the minimizer up to the $\pi/(p/2+1)$ sector is equivalent to the circle being the minimizer in the Euclidean plane with any perimeter density r^p, $p \in [0,1]$ and area density 1. As $p/2+1$ is the tangent line to $\sqrt{p+1}$ at 0, this is the best possible bound we could obtain that is linear in the denominator.

We now consider a more analytic formulation of the isoperimetric problem in the θ_0-sector with density r^p, and give an integral inequality that is equivalent to proving the conjectured angle of θ_1. The following integral inequality follows directly from the definitions of weighted area and perimeter.

We now consider a more analytic formulation of the isoperimetric problem in the θ_0-sector with density r^p, and give an integral inequality that is equivalent to proving the conjectured angle of θ_1.

Proposition 5.26. *In the θ_0-sector with density r^p, circles about the origin are isoperimetric if and only if the inequality*

$$[(p+1)\theta_0]^{\frac{1}{p+2}} \left[\int_0^{(p+1)\theta_0} r(\theta)^{\frac{p+2}{p+1}} d\theta \right]^{\frac{p+1}{p+2}} \leq \int_0^{(p+1)\theta_0} \sqrt{r^2 + \dot{r}^2} d\theta$$

holds for all C^1 functions $r(\theta)$.

Proof. Transition to Euclidean coordinates as in Proposition 5.3. For any polar graph $r(\theta)$ enclosed area and perimeter are given by:

$$A_r = c\frac{p+1}{p+2}\int_0^{(p+1)\theta_0} r^{\frac{p+2}{p+1}} d\theta,$$

$$P_r = \int_0^{(p+1)\theta_0} \sqrt{r^2 + \dot{r}^2} d\theta.$$

Therefore, a circle about the origin of radius b has

$$A_b = c\frac{(p+1)^2}{p+2}\theta_0 b^{\frac{p+2}{p+1}},$$

$$P_b = (p+1)\theta_0 b.$$

Equating the areas yields

$$b = [(p+1)\theta_0]^{\frac{-(p+1)}{p+2}} \left[\int_0^{(p+1)\theta_0} r^{\frac{p+2}{p+1}} d\theta\right]^{\frac{p+1}{p+2}}.$$

If circles are minimizers, P_r will have perimeter greater than or equal to P_b for any choice of r. By Proposition 3.11, any potential minimizer is smooth. The proposition follows. Alternatively, if the inequality holds for all C^1 functions, any function r not equal to the constant function will have perimeter greater than or equal to P_b, meaning circles about the origin are minimizers. □

Corollary 5.27. *In the θ_0-sector with density r^p, circles about the origin are isoperimetric if and only if the inequality*

$$\left[\int_0^1 r^{\frac{p+2}{p+1}} d\alpha\right]^{\frac{p+1}{p+2}} \leq \int_0^1 \sqrt{r^2 + \frac{\dot{r}^2}{[(p+1)\theta_0]^2}} d\alpha$$

holds for all C^1functions $r(\alpha)$.

Proof. In the inequality from Proposition 5.26, substitute $\alpha = \theta/(p+1)\theta_0$. The result follows immediately. □

Remark. This gives a nice analytic proof that circles about the origin minimize for $\theta_0 = \pi/(p+1)$; letting $\theta_0 = \pi/(p+1)$, we have

$$\left[\int_0^1 r^{\frac{p+2}{p+1}} d\alpha\right]^{\frac{p+1}{p+2}} \leq \int_0^1 \sqrt{r^2 + \frac{\dot{r}^2}{\pi^2}} d\alpha.$$

When $p = 0$, this corresponds to the isoperimetric inequality in the half-plane with density 1. As pointed out by Leonard Schulman of CalTech, the left hand side is nonincreasing as a function of p, meaning the inequality holds for all $p > 0$.

Corollary 5.28. *In the θ_0-sector with density r^p, circles about the origin are isoperimetric for $\theta_0 = \pi/\sqrt{p+1}$ if and only if the inequality*

$$\left[\int_0^1 r^q d\alpha\right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q-1)\frac{\dot{r}^2}{\pi^2}} d\alpha$$

holds for all C^1 fictions $r(\alpha)$ for $1 < q \leq 2$.

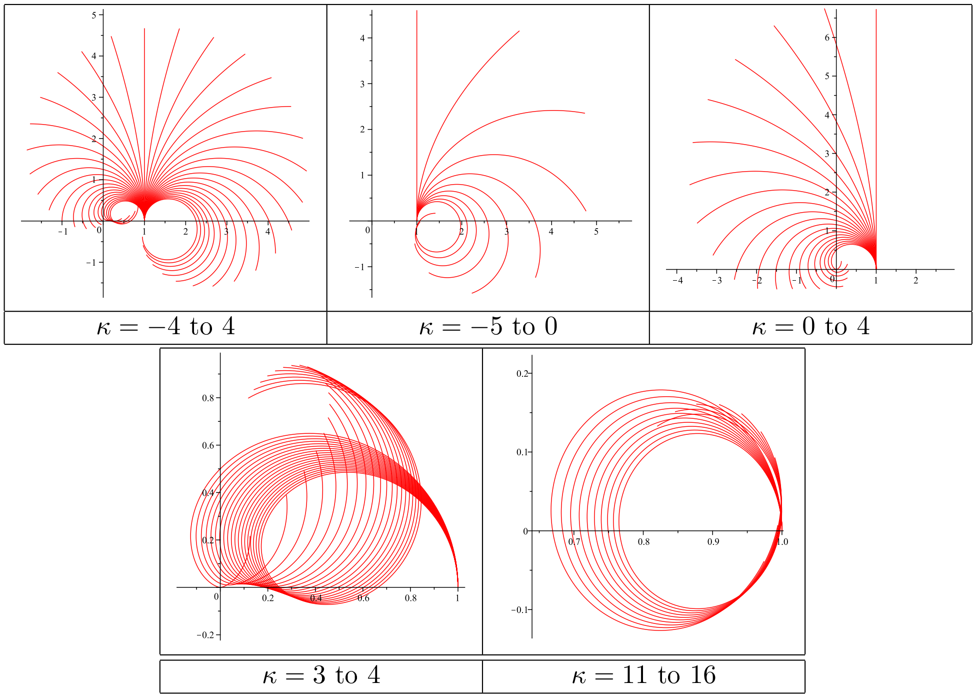


FIGURE 6.1. Constant curvature curves in Euclidean coordinates for p=1

Proof. In the inequality from Corollary 5.27, let $q = (p+2)/(p+1)$, and let $\theta_0 = \pi/\sqrt{p+1}$. □

Remark. We wonder if an interpolation argument might work here. When $q = 1$, the result holds trivially (equality for all functions r), and when $q = 2$, the inequality follows from the isoperimetric inequality in the half plane.

6. Constant Generalized Curvature Curves

We look at constant generalized curvature curves in greater depth. Proposition 6.14 proves that if the half period of constant generalized curvature curves is bounded above by $\pi(p+2)/(2p+2)$ and below by $\pi/\sqrt{p+1}$, then Conjecture 5.20 holds. Our major tools for studying constant generalized curvature curves are the second order constant generalized curvature equation and its first integral.

Lemma 6.1. *In the θ_0-sector with density r^p, an isoperimetric curve which is neither the circle nor the semicircle is not isoperimetric for any other sector.*

Proof. The curve has nonconstant, monotonic radius and attains its first critical point at θ_0 (Prop. 4.5 and Cor. 4.6). For any $\theta_1 < \theta_0$ the curve cannot hit the boundary perpendicularly and therefore cannot be isoperimetric (Prop. 4.4). Similarly, the curve is symmetric about θ_0 (Prop. 4.3), so for any $\theta_1 > \theta_0$ its radius cannot be monotonic, contradicting Corollary 4.6. □

Proposition 6.2. *In the θ_0-sector with density r^p, geodesics normal at the point $(1,0)$ are given by*

$$r(\theta) = (\sec((p+1)\theta))^{1/(p+1)}.$$

Proof. In Euclidean coordinates, generalized curvature is Riemannian curvature divided by the area density. Therefore, geodesics in Euclidean coordinates are lines, and the geodesic normal at $(1/(p+1), 0)$ is a vertical line given by $r(\theta) = \sec(\theta)/(p+1)$. Transition back to the original coordinates, and the proposition follows. When $p = 1$, this is a hyperbola through $(1,0)$. □

Proposition 6.3. *In the θ_0-sector with density r^p, a nonconstant curve $r(\theta)$ perpendicular to the point $(1,0)$ has constant generalized curvature λ if any only if it satisfies the differential equation*

$$\dot{r} = \pm r\sqrt{\frac{r^{2p+2}}{\left(\frac{\lambda}{p+2}\left(1 - r^{p+2}\right) - 1\right)^2} - 1}$$

until its first critical point. The positive solution is taken when $\lambda < p+1$, the negative when $\lambda > p+1$. Moreover, in Euclidean coordinates, a nonconstant curve with generalized curvature λ satisfies the differential equation

$$\dot{r} = \pm r\sqrt{\frac{r^2}{\left(\frac{(p+1)^{1/(p+1)}}{p+2}\lambda(1 - r^{(p+2)/(p+1)}) - 1\right)^2} - 1}.$$

Proof. Constant generalized curvature curves will be critical for the Lagrange multiplier functional

$$P - \lambda A = \int F = \int \left(r^p\sqrt{r^2 + \dot{r}^2} - \frac{\lambda}{p+2}r^{p+2}\right) d\theta.$$

We see $\lambda = dP/dA$, and thus is the generalized curvature. Since there is no explicit θ dependence in the integrand, we may use Beltrami's identity which says

$$\dot{r}\frac{\partial}{\partial \dot{r}}F - F = c$$

where c is a constant of integration. This gives:

$$\dot{r}\left(\frac{r^p\dot{r}}{\sqrt{r^2+\dot{r}^2}}\right) = c + r^p\sqrt{r^2+\dot{r}^2} - \frac{\lambda}{p+2}r^{p+2}$$

$$-r^{p+2} = \sqrt{r^2+\dot{r}^2}\left(c - \frac{\lambda}{p+2}r^{p+2}\right).$$

This simplifies to

$$\dot{r} = \pm r\sqrt{\frac{r^{2p+2}}{\left(c - \frac{\lambda}{p+2}r^{p+2}\right)^2} - 1}.$$

Set $r(0) = 1$ and $\dot{r}(0) = 0$, and the formula follows. $\lambda = p+1$ corresponds with the curvature of a circular arc, thus telling us when to take the positive or negative solution. The steps follow exactly in the Euclidean coordinates, except we now have

$$P - \lambda A = \int \sqrt{r^2+\dot{r}^2}d\theta - \lambda\frac{(p+1)^{1/(p+1)}}{p+2}\int r^{(p+2)/(p+1)}d\theta.$$

□

Remark. When dealing with this equation, we will slightly abuse notation and write $\dot{r} = ...$, which implies we are working with polar graphs. It turns out that constant generalized curvature curves outside the geodesic aren't actually polar graphs since they have tangent radial lines at isolated points. However, these curves still satisfy the differential equations above, with the stipulation that after the point where the curve fails to be a polar graph, we must replace $\dot{r} = +...$ with $\dot{r} = -....$ This detail is rather unimportant, though, since most often we are interested in radii where the value under the radicand is 0, which is independent of the sign out front. This equation can be used to generate pictures of constant generalized curvature curves, shown in Euclidean coordinates in Figure 6.1.

Remark. There is a better known second order equation for a curve with constant generalized curvature λ, given by Corwin *et al.* [CHHSX, Prop. 3.6]:

$$\ddot{r} = \frac{(p+1)r^2 + (p+2)\dot{r}^2 - \lambda(r^2+\dot{r}^2)^{3/2}}{r}.$$

The first order equation given above (in the θ_0-sector, not Euclidean coordinates), is actually the first integral of this second order equation.

Proposition 6.4. *In the θ_0-sector with density r^p, for each curve γ normal at $(1,0)$ with constant generalized curvature $\lambda \in \mathbb{R} - \{0, p+2\}$, there is a radius $r_1 \neq 1$ such that r is a critical radius of γ if and only if $r \in \{r_1, 1\}$. If $\lambda < p+1$, $r_1 > 1$; if $\lambda > p+1$, $r_1 < 1$.*

Proof. By Proposition 6.3, the radius r satisfies:

$$\dot{r} = \pm r\sqrt{\frac{r^{2p+2}}{\left(\frac{\lambda}{p+2}\left(1-r^{p+2}\right)-1\right)^2} - 1}.$$

If $\dot{r} = 0$, solving for λ yields

$$\lambda = (p+2)\frac{1 \pm r^{p+1}}{1 - r^{p+2}}.$$

The positive solution gives a bijection from $r \in [0,\infty)$ to $\lambda \in (-\infty, 0) \cup (p+2, \infty)$, whereas the negative solution gives a bijection from $r \in [0, \infty)$ to $\lambda \in (0, p+2)$. This means there is an inverse function from λ to r_1 in this range. Since constant generalized curvature curves are symmetric about critical points (Prop. 4.3), 1 and r_1 will be the two alternating values of critical points. Conversely, if $r = r_1$ or $r = 1$, we have $\dot{r} = 0$. Note that if $\lambda < p+1$, $r_1 > 1$, and if $\lambda > p+1$, $r_1 < 1$. □

Lemma 6.5. *In (r,θ)-space with density r^p, consider periodic, constant generalized curvature curves γ normal to the radial line $\theta = 0$. Reflection about the line $\theta = \alpha/2$ where α is angle at which γ attains its first critical value of radius followed by scaling by $1/\gamma(\alpha)$ gives a bijective correspondence between constant generalized curvature curves with curvature $0 < \lambda < p+1$ and constant generalized curvature curves with $p+1 < \lambda' = \lambda\gamma(\alpha) < p+2$ and between constant generalized curvature curves with curvature $\lambda < 0$ and $\lambda' = -\lambda\gamma(\alpha) > p+2$.*

Proof. Let $\gamma(\alpha) = r_1$. If $r_1 > 1$, γ corresponds with a constant generalized curvature curve γ_s normal at $(1,0)$ with final radius $1/r_1 < 1$, and vice versa. By scaling,

$\lambda' = \pm\lambda r_1$. The positive solution is taken for $0 < \lambda < p+1$; the negative for $\lambda < 0$. By Proposition 6.4,

$$\pm\lambda r_1 = \pm(p+2) r_1 \frac{1 \mp r_1^{p+1}}{1 - r_1^{p+2}}.$$

The positive solution maps bijectively onto $(p+1, p+2)$ as r_1 ranges from 1 to ∞, meaning curves with $0 < \lambda < p+1$ get mapped to curves with $p+1 < \lambda' < p+2$. The negative solution maps bijectively onto $(p+2, \infty)$ as r_1 ranges from 1 to ∞, meaning curves with $\lambda < 0$ get mapped to curves with $\lambda' > p+2$. □

Proposition 6.6. *In the θ_0-sector with density r^p, $p > 0$, curves normal at $(1,0)$ with constant generalized curvature $0 < \lambda < p+2$ do not cross each other before the curve with curvature further from the circle attains its maximum.*

Proof. By Lemma 6.5, we limit ourselves to $0 < \lambda < p+1$, the ranges of curvatures where the radius is increasing. Assume two constant generalized curvature curves r_1 and r_2 with curvatures $\lambda_1 > \lambda_2$ intersect each other at some angle α. Before the lower curvature curve attains its maximum, both curves have strictly positive derivative (Props. 4.5, 6.3), and initially $r_2 > r_1$. For them to cross at $r_1(\alpha) = r_2(\alpha) = r_\alpha$, we must have $\dot{r}_1(\alpha) > \dot{r}_2(\alpha)$. By Proposition 6.3, that implies

$$r_\alpha \sqrt{\frac{r_\alpha^{2p+2}}{\left(\frac{\lambda_1}{p+2}(1 - r_\alpha^{p+2}) - 1\right)^2} - 1} > r_1 \sqrt{\frac{r_\alpha^{2p+2}}{\left(\frac{\lambda_2}{p+2}(1 - r_\alpha^{p+2}) - 1\right)^2} - 1}.$$

Which implies $\lambda_1 < \lambda_2$, a contradiction. □

Proposition 6.7. *In the θ_0-sector with density r^p, constant generalized curvature curves normal at $(1,0)$ are periodic, except for the geodesic.*

Proof. Given the correspondence from Lemma 6.5, we work in the range $\lambda < p+1$. Given any curve other than the geodesic normal at $(1,0)$ with constant generalized curvature λ, there is a radius $r_1 > 1$ such that if the curve attains that radius, $\dot{r} = 0$ (Prop. 6.4). Since constant generalized curvature curves are symmetric about critical points (Prop. 4.3), the only way a polar graph could not be periodic is if it never reached r_1. Since $\dot{r}$ would have to approach zero, the radius would have to approach r_1. We recall that generalized curvature is defined as

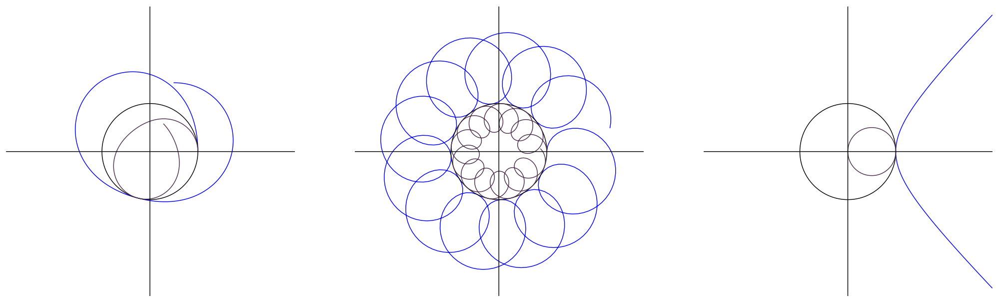
$$\lambda = \kappa - \frac{\partial\psi}{\partial n},$$

where κ is the Riemannian curvature and $\psi = p \log r$. Since $\dot{r}$ is approaching 0, $\partial\psi/\partial n$ must be approaching $-p/r_1$. However, since λ is constant and $\partial\psi/\partial n$ is approaching a constant, κ must approach the curvature of a circle with radius r_1. This implies

$$\lambda = \pm\frac{p+1}{r_1},$$

where the $\pm$ merely represents the oppositely oriented unit normal for $\lambda < 0$. Given our final radius r_1, we can find λ. From Proposition 6.4,

$$\lambda = (p+2)\frac{1 \pm r_1^{p+1}}{1 - r_1^{p+2}}.$$



(a) (b) (c)

FIGURE 6.2. Types of constant generalized curvature curves: a) for $0 < \lambda < p+2$ nonconstant periodic polar graphs (unduloids), b) for $\lambda < 0$ or $\lambda > p+2$ periodic nodoids, c) for $\lambda = p+2$, a circle through the origin, for $\lambda = p+1$ a circle about the origin, for $\lambda = 0$ a curve asymptotically approaching the radial lines $\theta = \pm\pi/(2p+2)$.

Equating these two expressions for λ yield

$$p+1 = \mp(p+2)r_1 \frac{1 \pm r_1^{p+1}}{1 - r_1^{p+2}}.$$

Taking the positive solution ($0 < \lambda < p+1$), L'Hopital's rule shows that as r_1 approaches 1, this equation is satisfied. However, the function on the right has derivative strictly greater than zero, so there are no other values of r_1 that satisfy this relationship. The negative solution ($\lambda < 0$) is ∞ when $r_1 = 1$ and asymptotically approaches $p+2$ from above, meaning this relationship can never be satisfied. Therefore constant curvature curves normal at $(1,0)$ must achieve their critical radii and are therefore periodic. □

Corollary 6.8. *For a curve $r(\theta)$ normal at the point $(1,0)$ with constant generalized curvature $0 \leq \lambda \leq 3$ in the θ_0-sector with density r^p, $p=1$, the value of r at its next critical point is:*

$$\frac{3 - \lambda + \sqrt{9 - 6\lambda - 3\lambda^2}}{2\lambda}.$$

Proof. By Proposition 6.4,

$$\lambda = 3\left(\frac{1-r^2}{1-r^3}\right)$$

provided $r \neq 0$. Expanding we find

$$-\lambda r^3 + \lambda - 3 + 3r^2 = 0.$$

Dividing through by $r-1$ and using the quadratic formula, the result follows immediately. □

The following two propositions classify constant generalized curvature curves in the plane with density r^p, as seen in Figure 6.2.

Proposition 6.9. *In the plane with density r^p, constant generalized curvature curves normal at $(1,0)$ with generalized curvature $0 < \lambda < p+2$ are periodic polar graphs.*

Proof. Given the correspondence from Lemma 6.5, we may assume $0 < \lambda < p+1$. In Euclidean coordinates, constant generalized curvature curves are convex (Prop. 5.3). Since the derivative is nonnegative before its first critical point after $(1,0)$ (Prop. 4.5) and the curve cannot cross the geodesic 6.6, the curve cannot go radial, and therefore remains a polar graph until its first critical point. By Proposition 6.7, the curve will attain this critical point, meaning it is a periodic polar graph. □

Proposition 6.10. *In the plane with density r^p, any curve r normal at $(1,0)$ with constant generalized curvature $\lambda < 0$ or $\lambda > p+2$ fail to be polar graphs.*

Proof. Given the correspondence from Lemma 6.5, we may assume $\lambda < 0$. In Euclidean coordinates, this curve has a critical point at some radius $r_1 > 1$ (Prop. 6.7). This implies that the curve is hitting some ray from the origin perpendicularly. However, since the curve is convex and starts outside the geodesic (Prop. 5.3), it must have crossed that ray once before, implying it is not a polar graph. □

Lemma 6.11. *In the θ_0-sector with density r^p, $p > 0$, an isoperimetric curve must have constant generalized curvature $0 < \lambda < p+2$.*

Proof. By Proposition 5.3, an isoperimetric curve is a polar graph. However, all curves with constant generalized curvature $\lambda < 0$ or $\lambda > p+2$ fail to be polar graphs (Prop. 6.10). When $\lambda = 0$, the curve is a line in Euclidean coordinates, which cannot meet both boundaries perpendicularly, contradicting regularity (Prop. 4.4). □

Proposition 6.12. *In the θ_0-sector with density r^p, curves normal at $(1,0)$ with constant generalized curvature $\lambda \approx p+1$ have half-period $T \approx \pi/\sqrt{p+1}$ with $T \geq \pi/\sqrt{p+1}$.*

Proof. Let r be a curve with constant generalized curvature $\lambda = p+1-\epsilon$. By Lemma 6.5 we may assume $\epsilon > 0$. By Proposition 6.7 we know r is periodic. By Proposition 6.3, r satisfies the second order equation:

$$\ddot{r} = \frac{(p+1)r^2 + (p+2)\dot{r}^2 - \lambda(r^2 + \dot{r}^2)^{3/2}}{r}.$$

Let $s_\lambda = (r-1)/\epsilon$. We note $s_\lambda(0) = 0$, and $\dot{s}_\lambda(0) = 0$. We may rewrite the equation above as:

$$\ddot{s}_\lambda = \frac{1}{1+\epsilon s_\lambda} \frac{(p+1)(1+\epsilon s_\lambda)^2 + (p+2)\epsilon^2 \dot{s}_\lambda^2 - \lambda((1+\epsilon s)_\lambda^2 + \epsilon^2 \dot{s}_\lambda^2)^{3/2}}{\epsilon}$$

We bound s_λ. We see $\epsilon s_\lambda = r - 1 < r_1 - 1$. From Proposition 6.4, we have

$$\lambda = (p+2)\frac{1 - r_1^{p+1}}{1 - r_1^{p+2}},$$

from which we see

$$\frac{d\lambda}{dr_1} \Big|_{r_1=1} = -\frac{1}{2}(p+1) < -\frac{1}{2},$$

implying

$$\left|\frac{dr_1}{d\lambda}\right| < 2,$$

so for small ϵ, $\epsilon s_\lambda < r - 1 < r_1 - 1 = 2\epsilon + O(\epsilon^2)$.

We now bound $\dot{s}_\lambda$. By Proposition 6.3, we have

$$\dot{r} = r\sqrt{\frac{r^{2p+2}}{\left(\frac{\lambda}{p+2}\left(1 - r^{(p+2)}\right) - 1\right)^2} - 1}.$$

Expanding about $r = 1$ by Taylor yields

$$\dot{r} = (1 + 2(p+1)(r-1))\sqrt{\frac{(1 + 2(p+1)(r-1) + O(\epsilon^2)}{(1 + (p+1-\epsilon)(r-1) + O(\epsilon^2))^2} - 1}$$

$$= (1 + O(\epsilon))\sqrt{\frac{1 + 2(p+1)(r-1) + O(\epsilon^2)}{1 + 2(p+1-\epsilon)(r-1) + O(\epsilon^2)} - 1} = O(\epsilon).$$

Thus we may write:

$$\ddot{s}_\lambda = \frac{1}{1+\epsilon s_\lambda}\frac{(p+1)(1 + 2\epsilon s_\lambda + O(\epsilon^2)) + O(\epsilon^2) - \lambda(1 + 2\epsilon s_\lambda + O(\epsilon^2))^{3/2}}{\epsilon}$$

$$= (1 + O(\epsilon))\,\frac{(p+1)(1 + 2\epsilon s_\lambda) + O(\epsilon^2) - \lambda(1 + 3\epsilon s_\lambda + O(\epsilon^2))}{\epsilon}$$

$$= (1 + O(\epsilon))\,\frac{p + 1 + 2(p+1)\epsilon s_\lambda + O(\epsilon^2) - (p+1) + \epsilon - 3(p+1-\epsilon)\epsilon s_\lambda}{\epsilon}$$

$$= (1 + O(\epsilon))\,(1 + 2(p+1)s_\lambda - 3(p+1-\epsilon)s_\lambda + O(\epsilon))$$

$$= 1 - (p+1)s_\lambda + O(\epsilon).$$

Note that $s_{p+1} = \left(1 - \cos\left(\sqrt{p+1}\,\theta\right)\right)/(p+1)$ satisfies the differential equation $\ddot{s}_{p+1} = 1 - (p+1)s_{p+1}$ and the initial conditions $s_{p+1}(0) = 0$ and $\dot{s}_{p+1}(0) = 0$, and is therefore the unique solution. s_{p+1} has half-period $\pi/\sqrt{p+1}$, which implies that for $\lambda \approx p+1$, s_λ has half-period $T \approx \pi/\sqrt{p+1}$. Finally, since the circle is stable for angles less than $\pi/\sqrt{p+1}$ (Prop. 5.18), we see $T \geq \pi/\sqrt{p+1}$. □

Proposition 6.13. *In the plane with density r^p, the half period T of an unduloid normal at $(1,0)$ with constant generalized curvature $0 < \lambda < p+1$ is given by*

$$T = \int_1^{r_1} \frac{dr}{r\sqrt{\frac{r^{2p+2}}{\left(\frac{1-r_1^{p+1}}{1-r_1^{p+2}}(1-r^{p+2})-1\right)^2} - 1}},$$

where $r_1$is the curve's first critical radius after $r = 1$.

Proof. This follows directly from the differential equation in Proposition 6.3. □

Remark. Our proof that curves with $\lambda \approx p+1$ have half periods $T \approx \pi/\sqrt{p+1}$ (Prop. 6.12) gives us a nice analytic result. That is, we see

$$\lim_{r_1 \to 1}\int_1^{r_1} \frac{dr}{r\sqrt{\frac{r^{2p+2}}{\left(\frac{1-r_1^{p+1}}{1-r_1^{p+2}}(1-r^{p+2})-1\right)^2} - 1}} = \frac{\pi}{\sqrt{p+1}}.$$

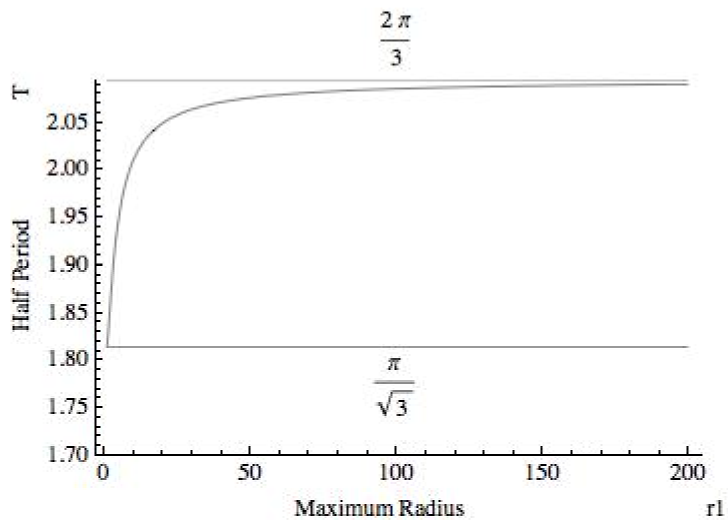


FIGURE 6.3. This Mathematica plot of unduloid half-period T as a function of maximum radius r_1 gives strong evidence for our main Conjecture 5.20 (see Prop. 6.14).

Proposition 6.14. *In the plane with density r^p, assume the half period of constant generalized curvature curves normal at $(1,0)$ with generalized curvature $p+1 < \lambda < p+2$ is bounded below by $\pi/\sqrt{p+1}$ and above by $\pi(p+2)/(2p+2)$. Then the conclusions of Conjecture 5.20 hold.*

Proof. By Proposition 5.6, minimizers are either circles about the origin, semicircles through the origin, or unduloids. By Lemma 6.1, unduloids can only minimize for one sector angle, namely their half period. Since the semicircle cannot minimize before $\pi(p+2)/(2p+2)$, and by assumption there are no unduloids with half period less than $\pi/\sqrt{p+1}$, the circle must be minimizing for all angles less than that. Similarly, since the circle cannot minimize after $\pi/\sqrt{p+1}$ and by assumption there are no unduloids with half period greater than $\pi(p+2)/(2p+2)$, the semicircle through the origin must be the minimizer for all angles greater than that. □

Remark. We use the equation from Proposition 6.13 to provide evidence for the bounds in Proposition 6.12. Figure 6.3 shows the Mathematica plot for $p = 2$, which is representative of all p. Moreover, the integral appears to be monotonic in r_1 which would imply that every unduloid with $0 < \lambda < p+2$ minimizes for exactly one θ_0-sector. Francisco López has suggested studying the integral above with the techniques of complex analysis.

7. The Isoperimetric Problem in Sectors with Disk Density

In this section we classify the isoperimetric curves in the θ_0-sector with density 1 outside the unit disk D centered at the origin and $a > 1$ inside D. Proposition 7.2 gives the potential minimizing candidates. Lemmas 7.4, 7.5, and 7.6 compare these candidates, and Theorems 7.8, 7.9, 7.10 classify the isoperimetric curves for every area and sector angle.

Definition 7.1. A *bite* is an arc of ∂D and another internal arc (inside D), the angle between them equal to $\arccos(1/a)$ (see Fig. 7.1(c)).

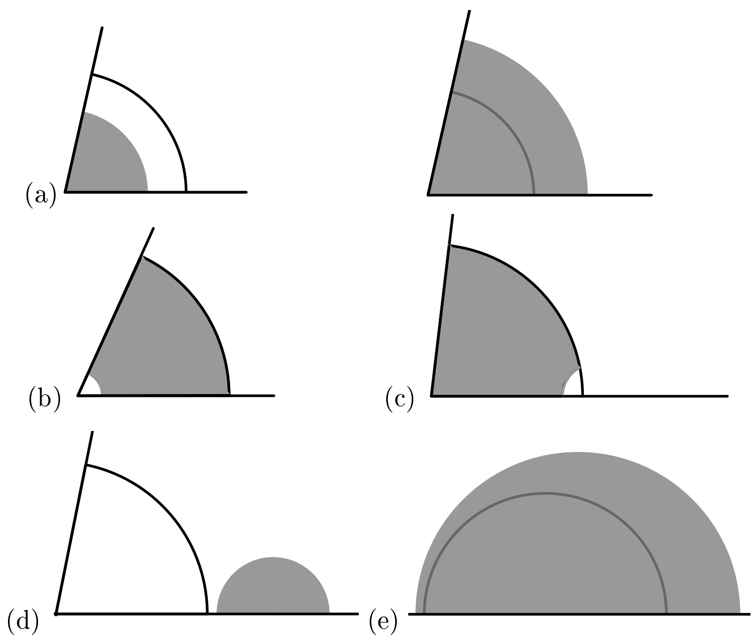


FIGURE 7.1. Isoperimetric sets in sectors with disk density: (a) an arc about the origin inside or outside D; (b) an annulus inside D; (c) a bite; (d) a semicircle on the edge disjoint from the interior of D; (e) a semicircle centered on the x-axis enclosing D for $\theta_0 = \pi$.

Proposition 7.2. *In the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, for area $A > 0$, an isoperimetric set is one of the following (see Fig. 7.1):*

(a) an arc about the origin inside or outside D;
(b) an annulus inside D with ∂D as a boundary;
(c) a bite;
(d) a semicircle on the edge disjoint from the interior of D;
(e) an arc centered on the x-axis enclosing D for $\theta_0 = \pi$.

Proof. Any component of a minimizer has to meet the edge normally by Proposition 4.4, since if not rotation about the origin brings it into contact with the edge or another component, contradicting regularity (Prop. 3.12). Since each part of the boundary has to have constant generalized curvature it must be made up of circular arcs. We can discard the possibility of combinations of circular arcs with the same density since one circular arc is better than n circular arcs. Therefore, there are five possibles cases:

(1) A circular arc from one boundary edge to itself (including possibly the origin).

There are three possibilities according to whether the semicircle has 0, 1, or 2 endpoints inside the interior of D. A semicircle with two endpoints inside D has an isoperimetric ratio of $2\pi a$. For a semicircle with one endpoint inside D (Fig. 7.2), by Proposition 3.12 Snell's Law holds. Then the only possible curve that intersects the boundary normally would also have to intersect D normally. Its perimeter and area satisfy:

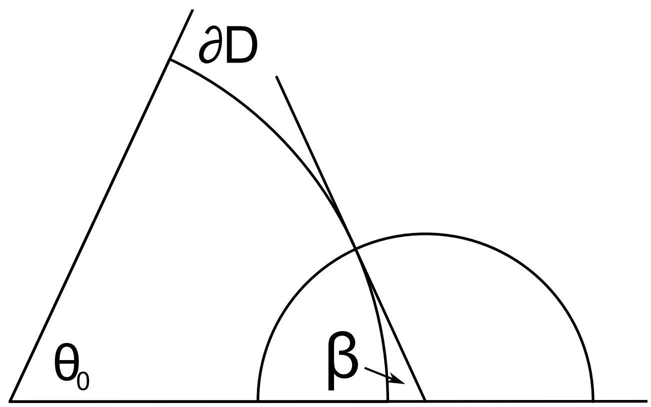


FIGURE 7.2. A semicircle meeting ∂D perpendicularly has more perimeter than a semicircle disjoint from the interior of D (Fig. 7.1d).

$$P = r(\pi - \beta + a\beta),$$
$$A = \frac{r^2}{2}\left(\pi - \beta + a\beta\right),$$
$$\frac{P^2}{A} = 2\left(\pi + \beta\left(a - 1\right)\right).$$

Therefore a semicircle (d) outside D with isoperimetric ratio 2π, is the only possibility.

(2) A circular arc from one boundary edge to another.
An arc (a) or annulus (b) about the origin are the only possibilities for any $\theta_0 < \pi$ or $A \leq a\theta_0/2$. At $\theta_0 = \pi$ and area $A > a\theta_0/2$ a semicircle (e) centered on the x-axis enclosing D is equivalent to a semicircle centered at the origin. For $\theta_0 > \pi$ and $A > a\theta_0/2$, a semicircle tangent to ∂D together with the rest of ∂D (Fig. 7.3) is in equilibrium, but we will show that it is not isoperimetric. Its perimeter and area satisfy:

$$P = \pi R + (\theta_0 - \pi), \qquad A = \frac{\pi}{2}\left(R^2 - 1\right) + \frac{\theta_0 a}{2}.$$

Comparing it with an arc (a) about the origin we see that it is not isoperimetric.

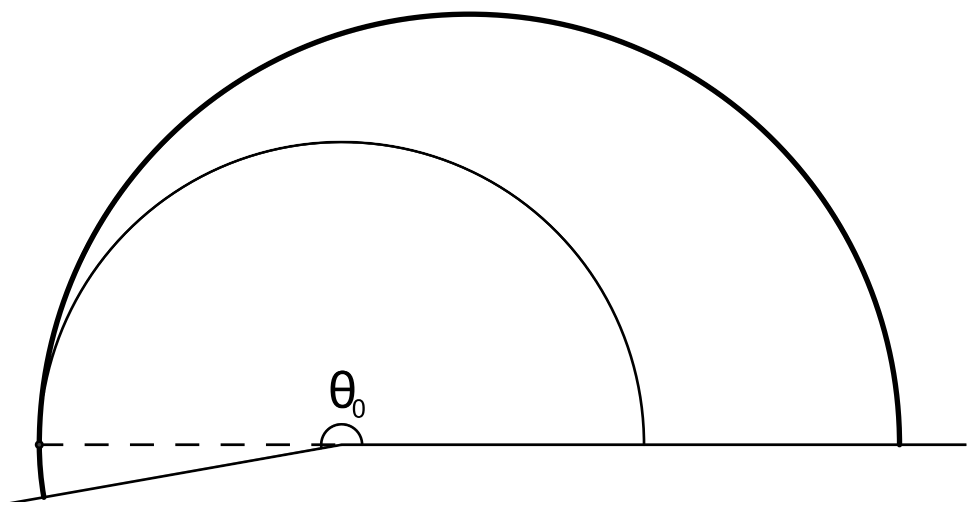


FIGURE 7.3. A semicircle tangent to ∂D together with the rest of ∂D is never isoperimetric.

(3) Two circular arcs (c) meeting along ∂D according to Snell's Law (Prop. 3.12).

(4) Three or more circular arcs meeting along ∂D.
By Cañete *et al.* [CMV, Prop. 3.19] this is never isoperimetric.

(5) Infinitely many circular arcs meeting along ∂D.
By Cañete *et al.* [CMV, Prop. 3.19] this is never isoperimetric.

□

Proposition 7.3. *For area $A \geq a\theta_0/2$, isoperimetric sets are one of the following (see Fig. 7.1):*
(a) semicircles on the edge disjoint from the interior of D;
(b) an arc about the origin;
(c) an arc centered at the x-axis enclosing D for $\theta_0 = \pi$.

Proof. By Proposition 7.2, isoperimetric sets are either (a), (d) or (e), since (b) and (c) don't enclose enough area. □

Lemma 7.4. *For area $A = a\theta_0/2$, the isoperimetric sets are (see Fig. 7.1):*
(1) if $\theta_0 < a\pi$, the boundary of the unit disk D;
(2) if $\theta_0 = a\pi$, both (1) and (3);
(3) if $\theta_0 > a\pi$, semicircles on the edge disjoint from the interior of D.

Proof. By Proposition 7.3 the isoperimetric sets for this area are either semicircles in the edge, ∂D or both. Comparing the isoperimetric ratios,

(a) for ∂D:

$$\frac{P^2}{A} = \frac{2\theta_0}{a},$$

(b) for the semicircle on the edge:

$$\frac{P^2}{A} = 2\pi.$$

The result follows. □

Lemma 7.5. *For*

$$\theta_1 = \arccos\frac{1}{a} - \frac{1}{a}\sqrt{1 - \frac{1}{a^2}}$$

and area A infinitesimally less than the area of D, in the θ_0-sector, isoperimetric sets are (see Fig. 7.1):

(1) if $\theta_0 < \theta_1$, an annulus inside D,

(2) if $\theta_0 = \theta_1$, both (1) and (3),

(3) if $\theta_0 > \theta_1$, a bite.

Proof. For area infinitesimally less than $a\theta_0/2$, semicircles (d) and arcs (a) cannot be isoperimetric since an annulus (b) and a bite (c) are better. Compare the ratio P^2/A of the region inside the first arc of the annulus and the region subtracted by the bite:

(a) for the region enclosed by the first arc of the annulus:

$$\frac{P^2}{A} = 2a\theta_0,$$

(b) for the region subtracted by the bite:

$$\frac{P^2}{A} = 2a\left(\arccos\frac{1}{a} - \frac{1}{a}\sqrt{1 - \frac{1}{a^2}}\right).$$

Equating the formulas we find that the annulus encloses more area with less perimeter if

$$\theta_0 < \theta_1 = \arccos\frac{1}{a} - \frac{1}{a}\sqrt{1 - \frac{1}{a^2}},$$

while if $\theta_0 > \theta_1$ the arc intersecting ∂D does better. Therefore, for areas infinitesimally less than that of the unit disk, it is better to subtract a small amount of area from the disk with an annulus if $\theta_0 < \theta_1$, but it is better to take out a bite if $\theta_0 > \theta_1$. The two tie when $\theta_0 = \theta_1$. □

Lemma 7.6. *In the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, for area $A > a\theta_0/2$ an isoperimetric curve is:*

(1) if $a\theta_0/2 < A < \theta_0^2(a-1)/2(\theta_0 - \pi)$, an arc about the origin;

(2) if $A = \theta_0^2(a-1)/2(\theta_0 - \pi)$, both (1) and (3);

(3) if $A > \theta_0^2(a-1)/2(\theta_0 - \pi)$, a semicircle on the edge disjoint from the interior of D.

Proof. By Proposition 7.3 an isoperimetric curve is one of the following: a semicircle on the edge disjoint from the interior of D, an arc about the origin, or both.

For a semicircle with radius r:

$$\frac{P^2}{A} = 2\pi.$$

For an arc about the origin with radius R:

$$\frac{P^2}{A} = \frac{2\theta_0 R^2}{a + R^2 - 1}.$$

By equating the ratios we calculate the values of R^2, r and area A_1 for which both ratios are equal:

$$R^2 = \frac{\pi(a-1)}{\theta_0 - \pi},$$
$$r = \theta_0\sqrt{\frac{(a-1)}{\pi(\theta_0 - \pi)}},$$
$$A_1 = \frac{\theta_0^2(a-1)}{2(\theta_0 - \pi)}.$$

Looking at the ratios, we see that isoperimetric curves are arcs about the origin for $A < A_1$, semicircles on the edge disjoint from the interior of D for $A > A_1$, and both for $A = A_1$. Moreover for $\theta_0 < \pi$, arcs about the origin are minimizers for any $A > a\theta_0/2$ and for $\theta_0 > a\pi$ semicircles are the minimizer for any $A > a\theta_0/2$ since A_1 never occurs in either case. □

Proposition 7.7. *In the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, for a fixed angle and area $A < a\theta_0/2$, once a bite encloses more area with less perimeter than an annulus inside D, it always does.*

Proof. Increasing the difference $a\theta_0/2 - A$, it will be sufficient to prove that a bite is better than a scaling of a smaller bite because an annulus changes by scaling. Therefore, there will be at most one transition.

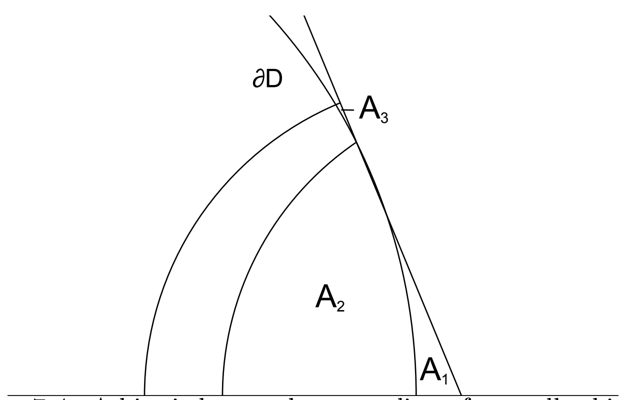


FIGURE 7.4. A bite is better than a scaling of a smaller bite, and hence once better than an annulus, it is always better.

To prove that a bite does better than a scaling of a smaller bite we are going to show that it has less perimeter and more area. Take a smaller bite and scale it by $1 + \epsilon$ (see Fig. 7.4) and eliminate the perimeter outside D. Another bite is created with perimeter P_a and area A_a:

$$P_a < (1+\epsilon)P, \qquad A_a = (1+\epsilon)^2 (A_1 + A_2) - A_3 > (1+\epsilon)^2 A_1,$$

Thus it is better than scaling a smaller bite. □

Theorem 7.8. *For some $\theta_2 < \pi$, in the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, for $\theta_0 \leq \pi$, there exists $0 < A_0 < A_1 < a\theta_0/2$, such that an isoperimetric curve for area A is (see Fig. 7.1):*

(1) if $0 < A < A_0$, an arc about the origin if $\theta_0 < \pi/a$, semicircles on the edge disjoint from the interior of D if $\theta_0 > \pi/a$, and both if $\theta_0 = \pi/a$;

(2) if $A = A_0$, both type (1) and (3);

(3) if $A_0 \leq A < A_1$ a bite, if $A_1 < A < a\theta_0/2$ an annulus inside D, and if $A = A_1$ both; if $\theta_0 > \theta_2$, $A_1 = a\theta_0/2$ and the annulus is never isoperimetric;

(4) if $A \geq a\theta_0/2$, an arc about the origin; at $\theta_0 = \pi$ any semicircle centered on the x-axis enclosing D.

Proof. By Proposition 7.2, there are only four sets that can be minimizers for areas less than $a\theta_0/2$. By Cañete *et al.* [CMV, Thm. 3.20] a bite is never isoperimetric for small areas since the semicircle is better. We compare the isoperimetric ratio of a semicircle on the edge disjoint from the interior of D, an arc about the origin and an annulus inside D:

for a semicircle:

$$\frac{P^2}{A} = 2\pi,$$

for an arc about the origin:

$$\frac{P^2}{A} = 2\theta_0 a,$$

for an annulus inside D with radius s:

$$\frac{P^2}{A} = \left(\frac{2\theta_0}{a}\right)\left(\frac{(1+as)^2}{1-s^2}\right).$$

For small areas, in case (c) , $s \to 1$ and the isoperimetric ratio increases. Equating (a) and (b), simple calculations show that minimizers are arcs about the origin for $\theta_0 < \pi/a$, semicircles for $\theta_0 > \pi/a$, and both for $\theta_0 = \pi/a$.

For area A infinitesimally less than $a\theta_0/2$, Lemma 7.5 shows the minimizers and the angle $0 < \theta_2 < \pi/2$ in which the transition occurs. Proposition 7.7 shows that once a bite encloses more area with less perimeter than an annulus, it always does. Then when the annulus starts minimizing for area infinitesimally less than $a\theta_0/2$ there can be at most one transition between it and a bite. Moreover if this transition occurs, there exists a $A_0 < A_1 < a\theta_0/2$ such that at the transition point area $A = A_1$ and $\theta_0 < \theta_2$. When a bite starts minimizing for area infinitesimally less than $a\theta_0/2$ it remains isoperimetric for every $A < a\theta_0/2$ until a semicircle in the edge or an arc about the origin beats it.

We can calculate the exact value of A_0 when the transition between an arc about the origin and an annulus inside D happens. Equating perimeter and area we conclude:

$$A_0 = \frac{\theta_0}{4}\left(a + \sqrt{2 - \frac{1}{a^2}}\right).$$

The same technique can be used to calculate A_0 when the transition between a semicircle in the edge and an annulus inside D happens. Then,

$$A_0 = \frac{\theta_0}{2} \frac{2\theta_0^2 a^2 + \left(\theta a^2 + \pi a\right)(\pi a - \theta_0) + 2\theta_0 a \sqrt{\theta_0^2 a^2 + \left(\theta a^2 + \pi a\right)(\pi a - \theta_0)}}{a\left(\theta_0 a + \pi\right)^2}.$$

For area $A = a\theta_0/2$, by Lemma 7.4, minimizers are arcs about the origin.

For area $A > a\theta_0/2$ and $\theta_0 \leq \pi$, by Lemma 7.6, minimizers are arcs about the origin. However, for $\theta_0 = \pi$, minimizers can be any arc centered at the x-axis and enclosing D since they are equivalent to a circular arc about the origin enclosing the same area.

Figure 7.5 shows transitions angle in function of density between arcs about the origin-semicircles in the edge and annulus inside D-a bite.

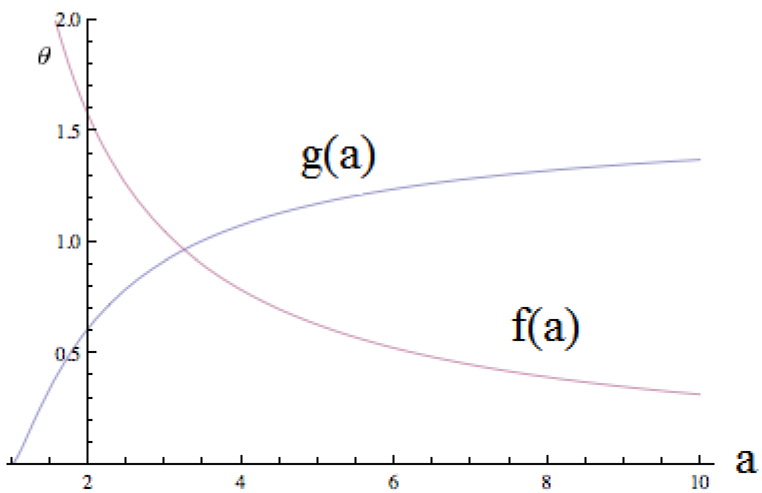


FIGURE 7.5. For $\theta_0 < f$ an arc about the origin is isoperimetric, while for $\theta_0 > f$ a semicircle in the edge is isoperimetric. For area infinitesimally less than $a\theta_0/2$, for $\theta_0 < g$ an annulus inside D is isoperimetric, while for $\theta_0 > g$ a bite is isoperimetric.

□

Theorem 7.9. *In the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, for $\pi < \theta_0 \leq a\pi$, there exists A_0, A_1, such that an isoperimetric curve for area A is (see Fig. 7.1):*

(1) if $0 < A < A_0$, a semicircle on the edge disjoint from the interior of D;
(2) if $A = A_0$, both type (1) and (3);
(3) if $A_0 < A < a\theta_0/2$, a bite;
(4) if $a\theta_0/2 \leq A < \theta_0^2(a-1)/2(\theta_0 - \pi)$, an arc about the origin;
(5) if $A = \theta_0^2(a-1)/2(\theta_0 - \pi)$, both type (4) and (6);
(6) if $A > \theta_0^2(a-1)/2(\theta_0 - \pi)$, a semicircle on the edge disjoint from the interior of D.

Proof. For cases (1), (2) and (3) Theorem 7.8 gives the minimizers. For area $A = a\theta_0/2$, Lemma 7.4 gives the minimizer. For cases (4) without the equality, (5) and (6), Lemma 7.6 gives the minimizers. □

Theorem 7.10. *In the θ_0-sector with density $a > 1$ inside the unit disk D and 1 outside, for $\theta_0 > a\pi$, the isoperimetric curves for area A are semicircles on the edge disjoint from the interior of D.*

Proof. Theorem 7.8 shows that an isoperimetric curve is a semicircle on the edge disjoint from the interior of D or a bite. As the area A approaches $a\theta_0/2$ from below, the perimeter P and area A of a bite satisfy:

$$P \to \theta_0,\ A \to a\theta_0/2,\ \frac{P^2}{A} \to \frac{2\theta_0}{a}.$$

So a semicircle on the edge disjoint from the interior of D, with ratio $P^2/A = 2\pi$, is isoperimetric for all $\theta_0 \geq a\pi$, and hence similarly for all $A \leq a\theta_0/2$. For area $A > a\theta_0/2$, Proposition 7.3 shows that an isoperimetric curve is one of the following: a semicircles on the edge disjoint from the interior of D or an arc about the origin. From Lemma 7.6 it follows that after $a\pi$, an arc about the origin cannot be isoperimetric. Therefore the minimizer for $\theta_0 > a\pi$ is a semicircles on the edge disjoint from the interior of D. □

8. Symmetrization

Proposition 8.3 extends a symmetrization theorem of Ros [R1, Sect. 3.2] to warped products with a product density as asserted by Morgan [M2, Thm. 3.2], and is general enough to include spherical symmetrization as well as Steiner and Schwarz symmetrization. Proposition 8.5 analyzes the case of equality in Steiner symmetrization after Rosales *et al.* [RCBM, Thm. 5.2]. Proposition 8.6 extends symmetrization to Riemannian fiber bundles with equidistant Euclidean fibers. Some simple examples are lens spaces (fibered by circles) as envisioned by Antonio Ros [R2, R1, Thm. 2.11], similar Hopf circle fibrations of $\mathbb{S}^{2n+1}$ over $\mathbb{CP}^n$, the Hopf fibration of $\mathbb{S}^7$ by great $\mathbb{S}^3$s and of $\mathbb{S}^{15}$ by great $\mathbb{S}^7$s. We were not, however, able to complete the proof by symmetrization envisioned by Vincent Bayle (private communication) of the conjecture that in $\mathbb{R}^n$ with a smooth, radial, log-convex density, balls about the origin are isoperimetric [RCBM, Conj. 3.12]. We thank Bayle and Ros for helpful conversations. Standard references on symmetrization are provided by Burago and Zalgaller [BZ, Sect. 9.2] and Chavel [C, Sect. 6]. Gromov [G, Sect. 9.4] provides some sweeping remarks and conditions for symmetrization, including fiber bundles.

Definition 8.1. The *Minkowski Perimeter* of a region R is

$$\liminf_{\delta r \to 0} \frac{\delta V}{\delta r}$$

for r enlargements. The limit exists and agrees with the usual definition of perimeter as long as the boundary of R is rectifiable and the metric and density are continuous (see [F, Thm 3.2.39]).

Lemma 8.2. (*Morgan* [M4, Lem. 2.4]) *Let f, h be real-valued functions on $[a,b]$. Suppose that f is uppersemicontinuous from the left and that h is C^1. Suppose that for $a \leq r < b$ the lower right derivative of f satisfies*

$$f'(r) = \liminf_{\Delta r \to 0} \frac{f(r+\Delta r) - f(r)}{\Delta r} \geq h'(r).$$

Then $f(b) - f(a) \geq h(b) - h(a)$.

Proposition 8.3. Symmetrization for warped products. *Let B be a Riemannian manifold. Consider a warped product $B \times \mathbb{R}^n$ with metric $ds^2 = db^2 + g(b)^2 dt^2$, with continuous product density $\varphi(b) \cdot \psi(t)$. Let R be a region of finite (weighted) perimeter. Suppose that in each fiber $\{b\} \times \mathbb{R}^n$ balls about the origin are isoperimetric. Then the Schwarz symmetrization* $\mathrm{sym}(R)$ *has the same volume and no greater perimeter than R.*

Remark. In the statement and proof $\mathbb{R}^n$ may be replaced by $\mathbb{S}^n$; also balls about the origin may be replaced by half planes $\{x_n \leq c\}$ (for $\mathbb{R}^{n-1} \times \mathbb{R}^+$ as well as $\mathbb{R}^n$) when these have finite weighted volume.

Proof. The preservation of volume is just Fubini's theorem.

For small r, denote r-enlargements in $B \times R$ by a superscript r and r-enlargements in fibers by a subscript r. Consider a slice $\{b_0\} \times C = R(b_0)$ of R and a ball about the origin $\{b_0\} \times D$ in the same fiber of the same weighted volume. For general b, consider slices $(\{b_0\} \times C)^r(b)$ of enlargements $(\{b_0\} \times C)^r$ of $\{b_0\} \times C$ and similarly slices $(\{b_0\} \times D)^r(b)$ of enlargements $(\{b_0\} \times D)^r$ of $\{b_0\} \times D$. We see that $(\{b_0\} \times C)^r(b) = \{b\} \times (C_{r'})$ and $(\{b_0\} \times D)^r(b) = \{b\} \times (D_{r'})$ for the same r' because g is invariant under vertical translation. Because the fiber density $\psi(t)$ is independent of b, $\{b\} \times C$ and $\{b\} \times D$ have the same weighted volume. Since every $\{b\} \times (D_{r'})$ is isoperimetric, by Lemma 8.2,

$$|\,\{b\} \times (D_{r'})| \leq |\,\{b\} \times (C_{r'})|,$$

and hence

$$(\{b_0\} \times D)^r(b) \subseteq \mathrm{sym}((\{b\} \times C)^r(b)).$$

Since this holds for all b,

$$(\{b_0\} \times D)^r \subseteq \mathrm{sym}((\{b_0\} \times C)^r).$$

Since this holds for all b_0,

$$(\mathrm{sym}(R))^r = \bigcup_{b_0 \in R} (\{b_0\} \times D)^r \subseteq \mathrm{sym}((\{b_0\} \times C)^r) \subseteq \mathrm{sym}(R^r).$$

Consequently,

$$|(\mathrm{sym}(R))^r| \leq |R^r|$$

and $sym(R)$ has no more perimeter than R, as desired. □

Remark. If $\psi(t)$ is a radial function, then Schwarz symmetrization may also be deduced from Steiner symmetrization by starting with Hsiang $\mathrm{SO}(n-1)$ symmetrization in $\mathbb{R}^n$ and modding out by $\mathrm{SO}(n-1)$ to reduce to Steiner symmetrization.

After Rosales *et al.* [RCBM, Thm. 5.2] we provide a uniqueness result for the case where the fibers are copies of $\mathbb{R}$, which is Steiner symmetrization. First we state a lemma of theirs. For a complete analysis of uniqueness without density or warping see Chlebík *et al.*[CCF].

Lemma 8.4. [RCBM, Lem. 5.3] *Suppose that we have finitely many nonnegative real numbers with $\sum_j \alpha_j a_j \geq 2\alpha a$ and $\sum_j \alpha_j \geq 2\alpha$. Then the following inequality holds*

$$\sum_j \alpha_j \sqrt{1 + a_j^2} \geq 2\alpha\sqrt{1 + a^2},$$

with equality if and only if $a_j = a$ *for every* j *and* $\sum_j \alpha_j = 2\alpha$.

Proposition 8.5. Uniqueness. *Let* ψ *be a smooth density on* $\mathbb{R}$ *such that centered intervals are uniquely isoperimetric for every prescribed volume. Let* B *be a smooth* n*-dimensional Riemannian manifold with density* φ*, and consider the warped product* $B \times_g \mathbb{R}$ *with metric* $ds^2 = db^2 + g(b)^2 dt^2$ *and product density* $\varphi(b) \cdot \psi(t)$*. Let* R *be an measurable set in* $B \times_g \mathbb{R}$ *such that almost every fiber intersects its topological boundary* ∂R *transversely where* ∂R *is locally an even number (possibly zero) of smooth graphs over* B *with nonvertical tangent planes, and let* R' *denote its Steiner symmetrization. Suppose that almost every fiber intersects* $\partial R'$ *where it is smooth with a nonvertical tangent plane or not at all, and that the fibers that don't intersect* $\partial R'$ *where it has a nonvertical tangent plane do not contribute anything to the perimeter. Suppose* R *and* R' *have the same perimeter. Then* $R = R'$ *up to a set of measure zero.*

Proof. Let D be the image of projection of R to B. Let $A \subseteq D$ be the set of points p in D for which ∂R and $\partial R'$ both have a nonvertical tangent planes above p. By the definition of Steiner symmetrization

$$\sum_{i\, odd} \int_{h_i}^{h_{i+1}} g(p)\varphi(p)\psi(x)dx = 2\int_0^{h^*} g(p)\varphi(p)\psi(x)dx$$

$$\sum_{i\, odd} \int_{h_i}^{h_{i+1}} \psi(x)dx = 2\int_0^{h^*} \psi(x)dx$$

on A where $h*$ is height function of $\partial R'$ with respect to B. Varying p we get that

$$\sum_{i\, odd} (\psi(h_{i+1})\nabla h_{i+1} - \psi(h_i)\nabla h_i) = 2\psi(h^*)\nabla h^*.$$

Hence

$$\sum_j \psi(h_i)|\nabla h_j| \geq 2\psi(h^*)|\nabla h^*|,$$

$$\sum_j \varphi(p)\psi(h_i)g(p)|\nabla h_j| \geq 2\varphi(p)\psi(h^*)g(p)|\nabla h^*|$$

on A. By the assumption that centered intervals are uniquely isoperimetric we have that

$$\sum_j \varphi(p)\psi(h_j) \geq 2\varphi(p)\psi(h^*),$$

with equality if and only if the corresponding slice of R is a centered interval.

By Lemma 8.4 with

$$\alpha_j = \varphi(p)\psi(h_j(p)),\ a_j = g(p)|\nabla h_j(p)|,\ \alpha = \varphi(p)\psi(h^*(p)),\ a = g(p)|\nabla h^*(p)|$$

we get

$$\sum_j \varphi(p)\psi(h_j(p))\sqrt{1 + g^2(p)|\nabla h_j(p)|^2} \geq 2\varphi(p)\psi(h^*(p))\sqrt{1 + g^2(p)|\nabla h^*(p)|^2},$$

with equality if and only if the number of graphs is two (or trivially 0), $g|\nabla h_1(p)| = g|\nabla h_2(p)| = g|\nabla h^*(p)|$ and the slice of R at p is a centered interval.

The perimeter $P(R)$ satisfies

$$P(R) \geq \int_A (\sum_j \varphi(p)\psi(h_j(p))\sqrt{1+g^2(p)|\nabla h_j(p)|^2})$$

$$\geq \int_A (2\varphi(p)\psi(h^*(p))\sqrt{1+g^2(p)|\nabla h^*(p)|^2})da = P(R')$$

because the fibers over $D - A$ contribute nothing to $P(R')$. If equality holds then R coincides with R' in almost every fiber of D, *i.e.*, $R = R'$ up to a set of measure 0. □

Remark. If for example we assume that ∂R and $\partial R'$ are smooth, then it follows that $R{=}R'$.

Now we want to prove a similar symmetrization theorem for fiber bundles that are "close enough" to warped products. So, consider a Riemannian fiber bundle $M \to B$ with equidistant Euclidean fibers $M_b = \mathbb{R}^n$ and such that parallel transport normal to the fibers from M_{b_1} to M_{b_2} scales the metric in the fibers by $g(b_2)/g(b_1)$. We can perform a symmetrization of a region R in M by taking a ball around the origin with the same volume as each slice of R in the warped product $B \times_g \mathbb{R}^n$. It's easy to see from the equidistance of fibers that this symmetrization will have the same volume, but it is a little trickier now to show that it has less perimeter. What we need is a way to compare locally the fiber bundle and the warped product. We accomplish this using parallel transport normal to the fibers.

We first observe that shortest paths in M from a point to a fiber are normal to the fibers: consider a shortest path from $p \in M$ to the fiber over $b \in B$. If we take only a portion of this path it must also be a shortest path from p to the fiber at its end point since if it were not, by the fact that fibers are equidistant we could extend a shortest path from p to this fiber to a path from p to the fiber over b that was shorter than our original path. The fact that fibers are equidistant also implies that at the end points of the shortest path between p and any fiber the tangent direction is orthogonal to the fibers. Combining these two facts we see that these paths of shortest distance between fibers are everywhere perpendicular to the fibers.

Now, consider $b_0 \in B$ and r such that the exponential map at b_0 is injective up to radius r. Then consider the open geodesic ball of radius r about the fiber over b_0, $M^r_{b_0}$. This is equal to the expansion of M_{b_0} by parallel transport normal to the fibers up to the radius r (indeed, if a point is in $M^r_{b_0}$ then its shortest path to M_{b_0} has length less than r and we know it is normal to the fibers everywhere).

Parallel transport normal to the fibers from b_0 to b scales the metric by $g(b)/g(b_0)$ and thus for each point in $M^r_{b_0}$ there is a unique point in M_{b_0} that maps to it under a parallel transport normal to the fibers. If we consider a shortest path from a point in M_{b_0} to another fiber in $M^r_{b_0}$ we see that it lies over a unique geodesic in B since the injectivity radius at b_0 is greater than r. So we see that for any $b \in B(b_0, r)$ there is a diffeomorphism γ_b from M_{b_0} to M_b given by parallel transport normal to the fibers over a unique geodesic in B that scales the metric by $g(b)/g(b_0)$.

If we fix an isometry j between M_{b_0} and $\mathbb{R}^n$ we obtain a diffeomorphism from $B(b_0, r) \times_g \mathbb{R}^n$ to $M^r_{b_0}$ by $f_r(b, y) = \gamma_b(j^{-1}(y))$. This is the map between the warped product and the fiber bundle given by normal parallel transport at b_0.

Proposition 8.6. Symmetrization for fiber bundles. *Consider a Riemannian fiber bundle $M \to B$ with equidistant Euclidean fibers $M_b = \mathbb{R}^n$ and a smooth positive function $g(b)$ such that parallel transport normal to the fibers from M_{b_1} to M_{b_2} scales the metric on the fibers by $g(b_2)/g(b_1)$. Suppose that M is compact or more generally that:*

1. *B is compact or more generally has positive injectivity radius and*
2. *for some $r_0 > 0$, for $r < r_0$ the r-tube about a fiber M_b under parallel translation from that fiber has metric $1+o(1)$ times that of a warped product $B(b,r) \times_g \mathbb{R}^n$, uniform in b.*

Let R be a region of finite perimeter. Consider the Schwarz symmetrization $\mathrm{sym}(R)$ *in the warped product $B \times_g \mathbb{R}^n$, which replaces the slice of R in each fiber with a ball about the origin of the same volume. Then* $\mathrm{sym}(R)$ *has the same volume and no greater perimeter than R.*

Remark. In the statement and proof the Euclidean fibers may be replaced by the other constant curvature model spaces: $\mathbb{S}^n$ or $\mathbb{H}^n$.

Proof. The preservation of volume follows from the equidistance of the fibers.

As in the proof of Proposition 8.3 denote r-enlargements in M by a superscript r and r-enlargements in fibers by a subscript r. Let r be a small positive number less than both r_0 and the injectivity radius of B. Consider a slice $C = R(b_0)$ of R and a ball D of the same volume about the origin in the corresponding fiber of $B \times_g R^n$. For general b, if C' denotes the image of C in M_b under normal parallel transport and D' denotes the copy of D in $\{b\} \times_g R^n$, $|C'| = |D'|$. As in the proof of Proposition 8.3, $|D^r(b)| = |D'_{r'}|$, but due to the twisting in fiber bundles, it is not necessarily true that $|C^r(b)| = |C'_{r'}|$. By the uniformity hypothesis (2), the map by parallel tranport based at M_{b_0} from $B \times_g \mathbb{R}^n$ to M distorts the metric by $1 + o(1)$, uniform over M. Therefore

$$C'_{r'} \subseteq C^{r+o(r)}(b).$$

Now, since each $D'_{r'}$ is isoperimetric, by Lemma 8.2,

$$|D'_{r'}| \le |C'_{r'}|.$$

And so we get

$$D^r(b) \subseteq \mathrm{sym}(C^{r+o(r)}(b)).$$

Since this holds for all b,

$$D^r \subseteq \mathrm{sym}(C^{r+o(r)}).$$

Since this holds for all b_0,

$$(\mathrm{sym}(R))^r = \bigcup_{b_0 \in R} D^r \subseteq \bigcup_{b_0 \in R} \mathrm{sym}(C^{r+o(r)}) \subseteq \mathrm{sym}(R^{r+o(r)}).$$

Consequently,

$$|(\mathrm{sym}(R))^r| \le |R^{r+o(r)}|,$$

and $\mathrm{sym}(R)$ has no more perimeter than R, as desired. □

In the special case where the metric in the fibers is preserved without rescaling we obtain a simpler statement and proof.

Theorem 8.7. Symmetrization for fiber bundles without warping. *Consider a Riemannian fiber bundle $M \to B$ with equidistant Euclidean fibers $M_b = \mathbb{R}^n$ and such that parallel transport normal to the fibers from M_{b_1} to M_{b_2} preserves the metric on the fibers. Suppose furthermore that the injectivity radius of B is bounded from below. Let R be a region of finite perimeter. Consider the Schwarz symmetrization* $\mathrm{sym}(R)$ *in the product $B \times \mathbb{R}^n$, which replaces the slice of R in each fiber with a ball about the origin of the same volume. Then* $\mathrm{sym}(R)$ *has the same volume and no greater perimeter than R.*

Proof. In this case distances on a point in a fixed fiber to points on a nearby fiber are no greater in the fiber bundle than in the product corresponding to a parallel transport from the fixed fiber. Indeed, consider a shortest path from the fixed fiber to a point on a nearby fiber in the product. Since the metric is the same in all fibers this path must lie over a geodesic in B. Its image in the fiber bundle consists of normal parallel transport corresponding to the horizontal motion in the product motion in the fiber corresponding to the vertical motion in the product. Thus the orthogonality of the vertical and horizontal motion is maintained. Furthermore the equidistance of fibers ensures the horizontal distance is maintained and the preservation of the metric in the fibers under normal parallel transport ensures that the vertical distance is maintained. So, we get a path of the same length in the fiber bundle. □

9. Isoperimetric Problems in $\mathbb{R}^n$ with Radial Density

In this section we look at the isoperimetric problem in $\mathbb{R}^n$ with a radial density. We use symmetrization (Prop. 8.3) to show that if a minimizer exists, then a minimizer of revolution exists (Lem. 9.1), thus reducing the problem to a planar problem (Lem. 9.2). We consider the specific case of $\mathbb{R}^n$ with density r^p and provide a conjecture (Conj. 9.3) and a nonexistence result (Prop. 9.5).

Lemma 9.1. *In $\mathbb{R}^n$ with a radial density if there exists an isoperimetric region then there exists an isoperimetric region of revolution.*

Proof. $\mathbb{R}^n$ is the same as $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ with warped product metric $ds^2 = dr^2 + r^2 d\Theta^2$. By Proposition 8.3 we can replace the intersection of the isoperimetric region with each spherical shell by a polar cap, preserving area without increasing perimeter. After performing this operation the region is rotationally symmetric about an axis, because each spherical cap is. □

Remark. This is just standard spherical symmetrization (see [C, Sect 6.4]).

Lemma 9.2. *The isoperimetric problem in $\mathbb{R}^n$ with density $f(r)$ is equivalent to the isoperimetric problem in the half plane $y > 0$ with density $y^{n-2} f(r)$.*

Proof. Up to a constant, this is the quotient space of $\mathbb{R}^n$ with density $f(r)$ modulo rotations about an axis. By Lemma 9.1, there exists a minimizer bounded by surfaces of revolution, so a symmetric minimizer in $\mathbb{R}^n$ corresponds to a minimizer in the quotient space and vice-versa. □

Isoperimetric curves in the plane with density r^p are circles about the origin when $p < -2$, do not exist when $-2 < p < 0$, and are circles through the origin when $p > 0$ ([CJQW, Props. 4.2, 4.3], [DDNT, Thm. 3.16]). We now look at the isoperimetric problem in $\mathbb{R}^n$ with density r^p.

Conjecture 9.3. *In $\mathbb{R}^n$ with density r^p isoperimetric regions are spheres about the origin if $p < -n$ and spheres through the origin if $p > 0$, and you can bound any volume with arbitrarily small area whenever $-n \leq p < 0$.*

Proposition 9.4. *In $\mathbb{R}^3$ with density r^p, $-3 < p < -2$, isoperimetric regions do not exist.*

Proof. Take a sphere centered at the origin, it has surface area cr^{p+2} and infinite volume outside the sphere. Therefore for any $\varepsilon > 0$ we can take two spheres centered at the origin each with surface area less than $\varepsilon/2$ that enclose any given volume between them. □

Remark. This argument is taken directly from [CJQW, Prop. 4.2] and holds for general $\mathbb{R}^n$ when $-n < p < -n + 1$.

Proposition 9.5. *For $-n < p < 0$ in $\mathbb{R}^n$ with density r^p, minimizers do not exist: you can enclose any volume with arbitrarily small perimeter.*

Proof. Consider a sphere S of radius $R \gg 0$ not containing the origin. Let r_{min}, r_{max} be the minimum and maximum values of r attained on S, note that $0 < r_{min} = r_{max} - 2R$. Let V, P be the volume and perimeter respectively of S. We have that:

$$V > c_0 R^n \min{}_S(r^p) = c_0 R^n r_{max}^p.$$

Fixing V we get that $r_{max} > c_1 R^q$ where $q = -n/p > 1$. This gives

$$r_{min} > c_1 R^q - 2R > \frac{c_1}{2} R^q$$

for sufficiently large R. Looking at P we have

$$P < c_2 R^{n-1} \max{}_S(r^p) = c_2 R^{n-1} r_{min}^p < c_2 R^{n-1} (\frac{c_1}{2} R^q)^p = c_3 R^{-1},$$

which can be made as small as we want with large enough R. □

Remark. A proof that applies for $-n < p < -n + 1$ was given by [CJQW, Prop. 4.2].

10. Computations with Maple

We offer numerical evidence for our conjectured transition angles (Conj. 5.20). Using Maple's ODE solver and the second order equation for constant curvature curves, we devised a program to search for constant generalized curvature curves with the best isoperimetric ratio P^{p+2}/A^{p+1}. The source code is available at the end of the section.

Consider the problem of finding an isoperimetric curve in a manner amenable to computer search. We want to restrict ourselves to as specific a situation as possible without losing any generality. Since this problem allows scaling (Prop. 5.7), we will fix a nonzero point on the angle 0 edge and consider only constant-curvature curves emanating from that edge perpendicularly at this point. Up to scaling any isoperimetric curve is equivalent to a curve in this family (Prop. 5.7). So, to find all isoperimetric curves all we need to do is examine all constant-curvature curves with a perpendicular intersection at this point and figure out which ones have the least ratio P^{p+2}/A^{p+1}. In fact we only have to examine such curves that are polar graphs (Prop. 5.3). As mentioned in the remark after Proposition 6.3, Corwin *et al.* [CHHSX, Prop. 3.6] give us the following formula:

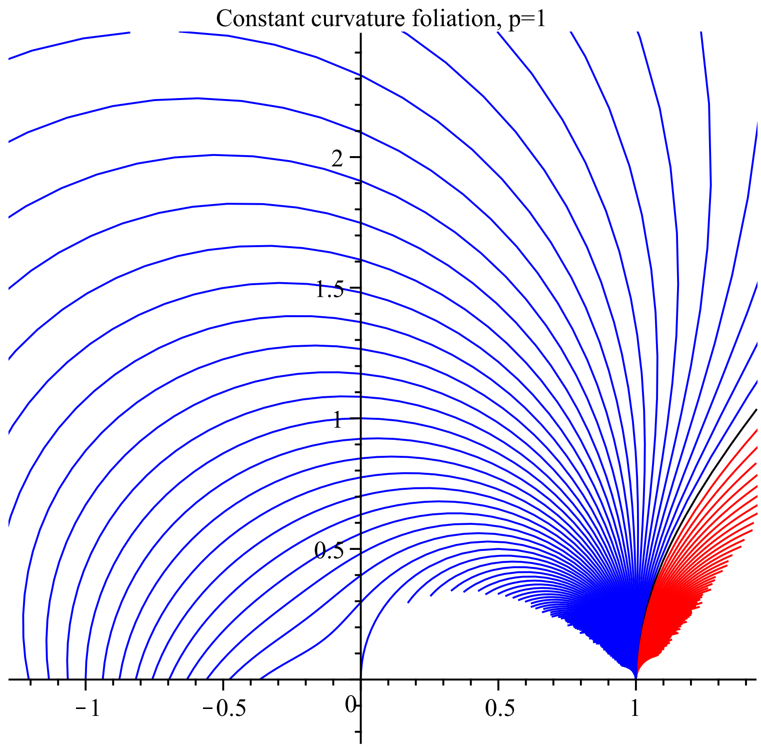


FIGURE 10.1. Constant curvature curves for $p = 1$ intersecting the radial line at angle 0 perpendicularly at radius 1. Negative curvatures appear in red, geodesic in black, and positive in blue; curves displayed for $-10 \leq \kappa \leq 20$.

Theorem 10.1. *In the plane with density r^p the following second order differential equation is satisfied by any polar graph $r(\theta)$ with constant generalized curvature λ:*

$$\lambda = \frac{(p+1)r^2 + (p+2)\dot{r}^2 - r\ddot{r}}{(r^2 + \dot{r}^2)^{3/2}}.$$

This also applies to constant generalized curvature curves in the θ_0-sector since any constant curvature curve there is a part of a constant generalized curvature curve in the plane. The restriction of perpendicularity (Prop. 4.4) gives us $\dot{r}(0) = 0$ and after fixing $r(0) = 1$ we have only one degree of freedom left, and so we get a one-parameter family of solutions given by varying λ. Using Maple's ODE solver we can get a picture of what these constant curvature curves look like. Figure 10.1 shows a representative set of them for $p = 1$, and this picture is characteristic of all $p > 0$.

What we would like to do is restrict ourselves to a reasonable range of curvatures to look at. Using the correspondence from Lemma 6.5 and the fact that minimizers must have $0 < \lambda < p + 2$ (Prop. 6.11), we see that it suffices to look between the circular arc and the semicircle, *i.e.* $p + 1 < \lambda < p + 2$.

This new reduced version of the problem is a calculation a computer can handle. Running it for several values of p, a solid pattern emerges. Up until $\pi/\sqrt{p+1}$, it

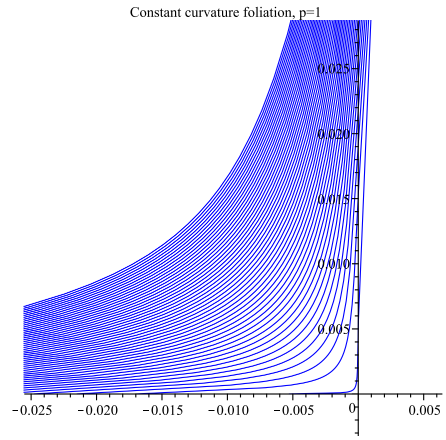


FIGURE 10.2. Constant curvature curves for $p = 1$ with curvature between 2.999 and 3.0

predicts the circular arc. Immediately thereafter there is a short transition period, after which it predicts the semicircle. This can be seen for $p = 1$ in Figure 1.2.

It has already been established that the circular arc has nonnegative second variation until exactly $\pi/\sqrt{p+1}$, which agrees with the program's prediction, and as we see no reason for the program to be producing unreliable data in this range we are reasonably confident that the circle minimizes up until this point. The program also accurately predicts that there will be a transition period between the circular arc and the semicircle, and in fact its prediction for the transition point between the unduloid and the semicircle fits in well with the conjectured angle of $\pi(p+2)/(2p+2)$. However, the length of this transition predicted by the program may be unreliable. To demonstrate why, we will first have to describe how the program functions.

First, it subdivides the curvature range into 100 parts. Then, for each of these curvatures it uses the Maple ODE solver to find a curve with this curvature meeting the other initial conditions $r(0) = 1$ and $\dot{r}(0) = 0$. Then, for each of these curves it uses numerical integration with about 400 sample points to calculate its weighted perimeter and area, and finally, tells us which one has the best ratio. We can make the number of divisions of either the curvature or the sampling points larger, but improvement in detail sees rapidly diminishing returns alongside rapidly increasing calculation time as demonstrated in Figure 10.2.

In Figure 10.2 there is only a .001 range of curvatures yet around the origin radii differ by about .05; that's 50 times as much. However, the program only uses increments of .01 curvature, approximately 500 times greater than the increments seen in Figure 10.2. Worse, in this picture as we approach the origin the spacing gets worse, so that even at this level of detail we still see a loss of discernment. In fact, there is no reason to look any closer, as at this level of detail the numeric error from the ODE solver is clearly visible − the curve that comes closest to the origin is actually the solver's version of the semicircle, which should be intersecting at the origin.

So, although the program provides very strong evidence that the circular arc minimizes up until the conjectured $\pi/\sqrt{p+1}$, it only suggests that the semicircle begins minimizing at the conjectured $\pi(p+2)/(2p+2)$. The source code is given below.

```
> PLow := 2 :
  PHigh := 3 :
  PSub := 2 :
  CurvatureLow := (p)→p + 1 :
  CurvatureHigh := (p)→p + 2 :
  CurvatureSub := 101 :
  AngleLow := 0 :
  AngleHigh := evalf(π·.999) :
  AngleSub := 401 : #this should be 1 mod 10...
  Data.dispose( ) :
  PositionData := Array(1 ..PSub, 1 ...CurvatureSub, 1 ..AngleSub) :
  RatioData := Array(1 ..PSub, 1 ..CurvatureSub, 1 ..AngleSub) :
> for i from 1 to PSub do:
```

if $PSub > 1$ then $p := PLow + \frac{(PHigh - PLow)}{PSub - 1}\cdot(i-1)$; else $p := PLow$; end if:

$$SectorArea := (r0, r1, \mathrm{T}) \rightarrow evalf\left(\int_0^T \int_0^{\left(\frac{t}{T}\cdot r1 + \left(1-\frac{t}{T}\right)\cdot r0\right)} r^{p+1}\,dr\,dt\right) :$$

$$SectorPerimeter := (\mathrm{r0, r1, T}) \rightarrow evalf\left(\int_0^T \left(\frac{t}{T}\cdot r1 + \left(1-\frac{t}{T}\right)\cdot r0\right)^p \cdot \sqrt{\left(\frac{t}{T}\cdot r1 + \left(1-\frac{t}{T}\right)\cdot r0\right)^2 + \left(\frac{(r1-r0)}{T}\right)^2}\,dt\right) :$$

for j from 1 to $CurvatureSub(p)$ do:

$$c := CurvatureLow(p) + \frac{(CurvatureHigh(p) - CurvatureLow(p))}{CurvatureSub - 1}\cdot(j-1);$$

$$EQ1 := \frac{d}{dt}r(t) = s(t) :$$

$$EQ2 := c = \frac{\left(r(t)^2 + 2\,s(t)^2 - r(t)\cdot\frac{d}{dt}s(t)\right)}{\left(r(t)^2+s(t)^2\right)^{\frac{3}{2}}} + \left(\frac{p}{\left(r(t)^2+s(t)^2\right)^{\frac{1}{2}}}\right) :$$

```
sol := dsolve({EQ1, EQ2, r(0) = 1, s(0) = 0}, numeric, range = AngleLow ..AngleHigh, output = listprocedure) :
getr := eval(r(t), sol) :
for k from 1 to AngleSub do:
```

$$ang := AngleLow + \frac{(AngleHigh - AngleLow)}{AngleSub - 1}\cdot(k-1) :$$

```
PositionData[i, j, k] := [ang, getr(ang)];
end do:
EQ1.dispose( ) :
EQ2.dispose( ) :
sol.dispose( ) :
getr.dispose( ) :

area := 0 :
```

```
perim := 0 :
anglechange := (AngleHigh − AngleLow)/(AngleSub − 1);
for k from 1 to AngleSub − 1 do:
area := area + SectorArea(PositionData[i, j, k][2], PositionData[i, j, k + 1][2], anglechange) :
perim := perim + SectorPerimeter(PositionData[i, j, k][2], PositionData[i, j, k + 1][2], anglechange) :
RatioData[i, j, k] := perim^(p + 2)/area^(p + 1) :
end do:

end do:
end do:

>
> Minimums = Array(1 ..PSub, 1 ..AngleSub − 1) :
for i from 1 to PSub do:
 for k from 1 to AngleSub − 1 do:
  least := [1, RatioData[i, 1, k]] :
  for j from 1 to CurvatureSub do:
  if RatioData[i, j, k] < least[2] then:
   least := [j, RatioData[i, j, k]]
  end if:
 end do:
 Minimums[i, k] := least;
end do:
end do:
> printlevel := 2 :
for i from 1 to PSub do:
 if PSub > 1 then p := PLow + (PHigh − PLow)/(PSub − 1) · (i − 1); else p := PLow end if:
Detail := 1 :
for k from 1 to (AngleSub − 1)/Detail do:


 ind := k · Detail :
angleat := AngleLow + (AngleHigh − AngleLow)/(AngleSub − 1) · (ind) :
 ToPlot := Array(1 ..ind) :
 curve := Minimums[i, ind][1] :
curvature := CurvatureLow(p) + (CurvatureHigh(p) − CurvatureLow(p))/(CurvatureSub − 1) · (curve − 1) :
ratio := Minimums[i, ind][2] :
 for s from 1 to ind do:
  ToPlot[s] := [PositionData[i, curve, s][2], PositionData[i, curve, s][1]] :
 end do:
fname := sprintf("f:/documents/small/results/temp/minimizer p=%f angle=%f curvature=%f.bmp", p, angleat, curvature) :
plottitle := sprintf("minimizer p=%f angle=%f curvature=%f, ratio=%f", p, angleat, curvature, ratio) :
 plotsetup(bmp, plotoutput = fname) :
 plot(convert(ToPlot,'list'), view = [−2 ..2,−2 ..2], title = plottitle, coords = polar) :
plotsetup(inline) :
end do;
end do;
```

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