1. Introduction

A density is a function weighting both perimeter and area. We study the isoperimetric problem on planar sectors with certain densities. The isoperimetric problem seeks to enclose prescribed (weighted) area with least (weighted) perimeter. Solutions are known for very few surfaces with densities (see Sect. 2 below). Our first major theorem after Dahlberg et al. [DDNT, Thm. 3.16] characterizes isoperimetric curves in a $\theta_0$-sector with density $r^p$, $p > 0$:

**Theorem.** (4.19) Given $p > 0$, there exist $0 < \theta_1 < \theta_2 < \infty$ such that in the $\theta_0$-sector with density $r^p$, isoperimetric curves are (see Fig. 1.1):

1. for $0 < \theta_0 < \theta_1$, circular arcs about the origin,
2. for $\theta_1 < \theta_0 < \theta_2$, unduloids,
3. for $\theta_2 < \theta_0 < \infty$, semicircles through the origin.

We give bounds on $\theta_1$ and $\theta_2$, but are unable to determine them exactly. Section 5 gives further results on constant generalized curvature curves. Sectors with density $r^p$ are related to $L^p$ spaces (see e.g. Cor. 4.27), have vanishing generalized Gauss curvature [CHHSX, Def. 5.1], and have an interesting singularity at the origin where density vanishes. Adams et al. [ACDLV] previously studied sectors with Gaussian density.

Our second major theorem after Cañete et al. [CMV, Thm. 3.20] characterizes isoperimetric curves in a $\theta_0$-sector with density $a > 1$ inside the unit disk and density 1 outside the unit disk. An interesting property of this problem is that it deals with a noncontinuous density. There are five different kinds of minimizers depending on $\theta_0$, $a$, and the prescribed area, shown in Figure 6.1.
Section 7 provides a general symmetrization theorem, including Steiner, Schwarz, and spherical symmetrization in products, warped products, and certain fiber bundles. Section 8 provides applications to $\mathbb{R}^n$ with radial densities.

1.1. The Sector with Density $r^p$. In the plane with density $r^p$, Carroll et al. [CJQW, Sect. 4] prove that for $p < -2$, isoperimetric curves are circles about the origin bounding area on the outside and prove that for $-2 \leq p < 0$, isoperimetric regions do not exist. Dahlberg et al. [DDNT, Thm. 3.16] prove that for $p > 0$, isoperimetric curves are circles through the origin. By a simple symmetry argument (Prop. 3.2), isoperimetric circles about the origin and circles through the origin in the plane correspond to isoperimetric circular arcs about the origin and semicircles through the origin in a $\pi$-sector. In this paper, we consider $\theta_0$-sectors for general $0 < \theta_0 < \infty$.

For $p \in (-\infty, -2) \cup (0, \infty)$, existence in the $\theta_0$-sector follows from standard compactness arguments (Prop. 2.4). Lemma 4.6 limits the possibilities to circular arcs about the origin, semicircles through the origin, and unduloids (nonconstant positive polar graphs with constant generalized curvature). Proposition 4.2 shows that if the circle is not uniquely isoperimetric for some angle $\theta_0$, it is not isoperimetric for all $\theta > \theta_0$. Corollary 4.14 shows that if the semicircle is ever minimizing, it is uniquely minimizing for all angles greater. Therefore, transitional angles $0 \leq \theta_1 \leq \theta_2 \leq \infty$ exist. Minimizers that depend on sector angle have been seen before, as in the characterization by Lopez and Baker [LB, Thm. 6.1, Fig. 10] of perimeter-minimizing double bubbles in the Euclidean cone of varying angles, which is equivalent to the Euclidean sector. Theorem 4.19 provides estimates on the values of $\theta_1$ and $\theta_2$.

We conjecture (Conj. 4.20) that $\theta_1 = \pi/\sqrt{p+1}$ and $\theta_2 = \pi(p + 2)/(2p + 2)$. Proposition 4.18 proves that the circle about the origin has positive second variation for all $\theta_0 < \pi/\sqrt{p+1}$, and Proposition 4.10 proves the semicircle through the origin is not isoperimetric for all $\theta_0 < \pi(p + 2)/(2p + 2)$. Numerics (Fig. 1.2) also support our conjecture.
An easy symmetry argument (Prop. 3.2) shows that the isoperimetric problem in the $\theta_0$-sector is equivalent to the isoperimetric problem in the $2\theta_0$-cone, a cone over $S^1$. We further note that the isoperimetric problem in the cone over $S^1$ with density $r^p$ is equivalent to the isoperimetric problem in the cone over the product of $S^1$ with a $p$-dimensional manifold $M$ among regions symmetric under a group of isometries acting transitively on $M$. This provides a classical interpretation of the problem, which we use in Proposition 4.9 to obtain an improved bound for $\theta_1$ in the $p = 1$ case by taking $M$ to be the rectangular two-torus.

1.2. Constant Generalized Curvature Curves. Section 5 provides further details on constant generalized curvature curves in the sector with density $r^p$, which are of interest since minimizers must have constant generalized curvature (see Sect. 2). Proposition 5.2 shows that if the unduloid periods are bounded above and below by certain values, then we can determine the exact values of $\theta_1$ and $\theta_2$ (see Conj. 4.20).

1.3. The Sector with Disk Density. Section 6 considers a sector of the plane with density $a > 1$ inside the unit disk and $1$ outside. Cañete et al. [CMV, Sect. 3.3] consider this problem in the plane, which is equivalent to the $\pi$-sector. Proposition 6.2 gives the five possibilities of Figure 6.1. Our Theorems 6.3, 6.4, and 6.5 classify isoperimetric curves in a $\theta_p$-sector, depending on $\theta_0$, density $a$, and area.

1.4. Symmetrization. In Section 7, we provide two general symmetrization theorems in arbitrary dimension and codimension, in products, warped products, and certain fiber bundles, including Steiner, Schwarz, and spherical symmetrization. Proposition 7.3 extends a symmetrization theorem of Ros [R1, Sect. 3.2] to warped
products with a product density, and is general enough to include spherical symmetrization as well as Steiner and Schwarz symmetrization. Proposition 7.6 extends symmetrization to Riemannian fiber bundles with equidistant Euclidean fibers such as certain lens spaces.

1.5. \( \mathbb{R}^n \) with Radial Density. Section 8 considers the isoperimetric problem in \( \mathbb{R}^n \) with radial density. We use spherical symmetrization to reduce the problem to a two dimensional isoperimetric problem in a plane with density. We specifically consider \( \mathbb{R}^n \) with density \( r^p \) and provide a conjecture (Conj. 8.3) and a nonexistence result (Prop. 8.4).

1.6. Open Questions.

1. How can one prove Conjecture 4.20 on the values of the transitional angles \( \theta_1 \) and \( \theta_2 \)?
2. Could the values of \( \theta_1 \) and \( \theta_2 \) be proven numerically for fixed \( p \)?
3. If a circular arc about the origin is isoperimetric in the \( \theta_\theta \)-sector with density \( r^p \), is it isoperimetric in the same \( \theta_\theta \)-sector with density \( r^q \), \( q < p \)?
4. Are circles about the origin isoperimetric in the Euclidean plane with perimeter density \( r^p \), \( p \in (0,1) \)? (See Rmk. after Conj. 4.25.)
5. In the plane with density \( r^p \), do curves with constant generalized curvature near that of the semicircle have half period \( T \approx \pi (p + 2) / (2p + 2) \)? (See Conj. 4.20 and Rmk. 5.3 for the corresponding result near the circular arc.)
6. Are spheres through the origin isoperimetric in \( \mathbb{R}^n \) with density \( r^p \), \( p > 0 \)? (See Conj. 8.3.)

1.7. Acknowledgments. This paper is the work of the 2009 SMALL Geometry Group in Granada, Spain. We would like to thank the National Science Foundation and Williams College for funding both SMALL and our trip to Granada. We would also like to thank the University of Granada’s Department of Geometry and Topology for their work and support. In particular, we thank Rafael López for opening his home and his office to us. We would like to thank Antonio Cañete, Manuel Ritoré, Antonio Ros, and Francisco López for their comments. We thank César Rosales for arranging our wonderful accommodations at Carmen de la Victoria. We thank organizers Antonio Martínez, José Antonio Gálvez, and Francisco Torralbo of the Escuela de Análisis Geométrico. Leonard J. Schulman of CalTech, Casey Douglas of St. Mary’s College, and Gary Lawlor of BYU all provided helpful insights during our research. Finally, we would like to thank our research advisor Frank Morgan, without whom this paper would never have been able to take shape.

2. Isoperimetric Problems in Manifolds with Density

A density on a Riemannian manifold is a nonnegative, lower semicontinuous function \( \Psi(x) \) weighting both volume and hypersurface area. In terms of the underlying Riemannian volume \( dV_0 \) and area \( dA_0 \), the weighted volume and area are given by \( dV = \Psi dV_0 \), \( dA = \Psi dA_0 \). Manifolds with densities arise naturally in geometry as quotients of other Riemannian manifolds, in physics as spaces with different mediums, in probability as the famous Gauss space \( \mathbb{R}^n \) with density \( \Psi = ce^{-a^2r^2} \), and in a number of other places as well (see Morgan [M1, Ch. 18, M5]).
The generalized mean curvature of a manifold with density $\Psi(x) = e^{\psi(x)}$ is defined to be

$$H_\psi = H - \frac{1}{n - 1} \frac{d\psi}{dn},$$

where $H$ is the Riemannian mean curvature, as this corresponds to the first variation of weighted area ([M5, Intro.], [M1, Ch. 18]). In two dimensions, the focus of this paper, this reduces to

$$\lambda = \kappa - \frac{d\psi}{dn},$$

where $\kappa$ is the Riemannian curvature.

The isoperimetric problem on a manifold with density seeks to enclose a given weighted area with the least weighted perimeter. As in the Riemannian case, for a smooth density isoperimetric hypersurfaces have constant generalized curvature. The solution to the isoperimetric problem is known only for a few manifolds with density including Gauss space (see [M1, Ch. 18]) and the plane with a handful of different densities (see Cañete et al. [CMV, Sect. 3], Dahlberg et al. [DDNT, Thm 3.16], Engelstein et al. [EMMP, Cor. 4.9], Rosales et al. [RCBM, Thm. 5.2], and Maurmann and Morgan [MM, Cor. 2.2]). Existence and regularity are discussed in our report [DHHT, Sect. 3], and the major results are given here.

**Proposition 2.1.** [CJQW, Prop. 4.2] In the circular cone with density $r^p$ for $p \in (-2, 0)$ no isoperimetric regions exist.

*Proof.* This is proved in the plane by constructing curves of arbitrarily low perimeter bounding any area. Their argument extends immediately to the sector. \qed

The next theorem gives a general existence condition on isoperimetric surfaces in manifolds with area density which, after a change of coordinates, includes the circular cone with density $r^p$.

**Theorem 2.2.** Suppose $M^n$ is a smooth connected possibly non-complete $n$-dimensional Riemannian manifold with isoperimetric function $I$ such that $\lim_{A \to \infty} I(A) = \infty$. Suppose furthermore that every closed geodesic ball of finite radius in $M$ has finite volume and finite boundary area and there is some $C \in \mathbb{Z}^+$ and $x_0 \in M$ such that the complement of any closed geodesic ball of finite radius about $x_0$ contains finitely many connected components and at most $C$ unbounded connected components. Then for standard boundary area and any lower semi-continuous positive volume density $f$ such that

1. for some $x_0 \in M^n$, $\sup \{f(x) \mid \text{dist}(x, x_0) > R \}$ goes to 0 as $R$ goes to $\infty$;
2. for some $\epsilon > 0$, $\{x \mid B(x, \epsilon) \text{ is not complete} \}$ has finite weighted volume;

isoperimetric regions exist.

*Proof.* Using standard compactness arguments we see it suffices to show that no weighted volume can escape outside of an increasing sequences of compact sets whose union is $M^n$. That no weighted volume can escape to points of noncompleteness is immediate from the second condition on the density. To show that none can escape to infinity we first isolate a finite number of “ends” of the manifold and then show that in each of these no weighted volume can disappear either because there is only finite weighted volume there or because we can use the standard isoperimetric inequality combined with the density approaching 0. For details see [DHHT, Thm. 3.9]. \qed
We state now a regularity result by Morgan [M6, Cor. 3.8, Sect. 3.10].

**Theorem 2.3.** For $n \leq 6$, let $S$ be an $n$-dimensional isoperimetric hypersurface in a manifold $M$ with smooth Riemannian metric and smooth positive volume and perimeter densities. Then $S$ is a smooth submanifold.

The next two propositions give the existence and regularity of minimizers in the main cases that we examine in the rest of the paper.

**Proposition 2.4.** In the circular cone with density $r^p$ for $p \in (-\infty, -2) \cup (0, \infty)$ isoperimetric curves exist and are smooth.

**Proof.** Through the change of coordinates $w = z^{p+1}/(p+1)$, the circular cone with density $r^p$ is equivalent to a circular cone with Euclidean perimeter and area density $r^q$ for $q \in (-2, 0)$. Furthermore it suffices to consider the cone with deleted vertex. Existence follows by Theorem 2.2. Smoothness follows from Theorem 2.3. \qed

**Proposition 2.5.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, isoperimetric curves exist for any given area. These isoperimetric curves are smooth except at the boundary of $D$, where they obey the following Snell refraction rule (see Fig. 2.1):

$$\frac{\cos \alpha_+}{\cos \alpha_-} = \frac{1}{a},$$

where $\alpha_+$ is the angle of intersection from inside of $D$ and $\alpha_-$ is the angle of intersection from outside of $D$.

**Proof.** Cañete et al. [CMV, Thm. 3.18] prove existence for the plane. The same result and proof hold for the $\theta_0$-sector. The regularity and Snell refraction rule follow from [CMV, Prop. 2.14]. \qed
3. Isoperimetric Regions in Sectors with Density

We study properties of isoperimetric regions in planar sectors with radial densities. Proposition 3.2 shows there is a one-to-one correspondence between minimizers in the $\theta_0$-sector and in the $2\theta_0$-cone, modulo rotations. Propositions 3.4 and 3.5 provide some regularity results.

**Lemma 3.1.** Given an isoperimetric region in the $2\theta_0$-cone with density $f(r)$, there exist two rays from the origin separated by an angle of $\theta_0$ that divide both the area and perimeter of the region in half.

**Proof.** First we show that there are two such rays that separate the area of the region in half. Take any two rays from the origin separated by an angle of $\theta_0$. Rotate them around, keeping an angle of $\theta_0$ between them. The area between them varies continuously as we rotate the rays, and thus so does their difference, $A$. By the time we rotate the rays by an angle of $\theta_0$, $A$ has changed to $-A$. By the intermediate value theorem, at some point the difference must be 0, implying there are two rays separated by an angle $\theta_0$ that divide the area in half. If one side had less perimeter than the other, we could reflect it to obtain a region with the same area and less perimeter than our original region, violating the condition that it was isoperimetric. □

**Proposition 3.2.** An isoperimetric region of area $2A$ in the cone of angle $2\theta_0$ with density $f(r)$ has perimeter equal to the twice the perimeter of an isoperimetric region of area $A$ in the sector of angle $\theta_0$ with density $f(r)$. Indeed, the operation of doubling a sector to form a cone provides a one-to-one correspondence between isoperimetric regions in the sector and isoperimetric regions in the cone, modulo rotations.

**Proof.** Given an isoperimetric region in the sector, take its reflection into the cone to obtain a region in the cone with twice the area and twice the perimeter. This region must be isoperimetric for the cone, for if there were a region in the cone with less perimeter for the same area we could divide its area in half by two rays separated by an angle $\theta_0$ as in Lemma 3.1, take the side with at most half the perimeter to obtain a region in the $\theta_0$-sector with the same area and less perimeter than our original isoperimetric region.

Conversely, given an isoperimetric region in the cone, divide its area and perimeter in half by the two rays described in Lemma 3.1. Both regions must be isoperimetric in the sector, for if there were a region with less perimeter for the same area, taking its double would yield a region in the cone with less perimeter than our original region for the same area. □

We shall find many occasions to use the following proposition of Dahlberg et al.

**Proposition 3.3.** [DDNT, Lem. 2.1] Consider $\mathbb{R}^2 - \{0\}$ with smooth radial density $e^{\psi(r)}$. A constant generalized curvature curve is symmetric under reflection across every line through the origin and a critical point of $\psi$.

**Proof.** This holds from the uniqueness of ordinary differential equations. We note that the same proof holds in a sector. □

**Proposition 3.4.** In the $\theta_0$-sector with smooth density $f(r)$, isoperimetric curves meet the boundary perpendicularly.
Proof. By Proposition 3.2 there is a one-to-one correspondence between minimizers in the sector and the cone, meaning the double of this curve in the cone of angle $2\theta_0$ is minimizing and hence smooth (Prop. 2.4). Therefore, the original curve meets the boundary perpendicularly.

**Proposition 3.5.** In the $\theta_0$-sector with smooth density $f(r)$, if isoperimetric curves are nonconstant polar graphs, they do not contain a critical point on the interior.

**Proof.** Assume there is an isoperimetric curve $r$ with a critical point on the interior. By Proposition 3.4, $\dot{r}(0) = \dot{r}(\theta_0) = 0$. Since constant generalized curvature curves are symmetric under reflection across a line through the origin and a critical point of $r$ (Prop. 3.3), in the cone of angle $2\theta_0$ this curve has at least four critical points. By symmetry, critical points must be strict extrema. Let $C$ be a circle about the axis intersecting the curve in at least four points. $C$ divides the curve into at least two regions above $C$ and two regions below $C$. Interchanging one region above $C$ with a region below $C$ results in a region with the same perimeter and area whose boundary is not smooth. Since isoperimetric curves must be smooth, $r$ cannot be a minimizer.

**Corollary 3.6.** If an isoperimetric curve $r(\theta)$ in the $\theta_0$-sector with smooth density is a nonconstant polar graph, $r$ must be strictly monotonic.

**Remark.** Proposition 3.5 and Corollary 3.6 strengthen some arguments of Adams et al. [ACDLV, Lem. 3.6].

4. The Isoperimetric Problem in Sectors with Density $r^p$

Our main theorem, Theorem 4.19, characterizes isoperimetric regions in a planar sector with density $r^p$, $p > 0$. The subsequent results consider an analytic formulation of the problem.

**Proposition 4.1.** In the half plane with density $r^p$, $p > 0$, semicircles through the origin are the unique isoperimetric curves.

**Proof.** By Proposition 3.2, there is a one-to-one correspondence between minimizers in the $\theta_0$-sector and the $2\theta_0$-cone; in particular there is a correspondence between minimizers in the half plane and minimizers in the $2\pi$-cone, i.e., the plane. Since circles through the origin are the unique minimizers in the plane (see Sect. 1.1), semicircles through the origin are the unique minimizers in the half plane.

**Proposition 4.2.** For density $r^p$, if the circle about the origin is not uniquely isoperimetric in the $\theta_0$-sector, for all $\theta > \theta_0$ it is not isoperimetric.

**Proof.** Let $r$ be a non-circular isoperimetric curve in the $\theta_0$-sector, and let $C$ be a circle bounding the same area as $r$. For any angle $\theta > \theta_0$, transition to the $\theta$-sector via the map $\alpha \rightarrow \alpha \theta/\theta_0$. This map multiplies area by $\theta/\theta_0$, and scales tangential perimeter. Therefore, if $r$ had the same or less perimeter than $C$ in the $\theta_0$-sector, its image under this map has less perimeter than a circle about the origin in the $\theta$-sector.

**Proposition 4.3.** In the $\theta_0$-sector with density $r^p$, $p > 0$, an isoperimetric region contains the origin, and its boundary is a polar graph.
Proof. Work in Euclidean coordinates via the mapping \( w = z^{p+1}/(p+1) \). Here perimeter is Euclidean perimeter and the area density is \( cw^{-q} \) where \( q = p/(p+1) \). Since the area density is strictly decreasing away from the origin, an isoperimetric region must contain the origin. Any minimizer is bounded by a smooth curve of constant generalized curvature (Thm. 2.3), and since generalized curvature is just Riemannian curvature divided by the area density [CJQW, Def. 3.1], the Riemannian curvature does not change sign, and the curve is convex. Thus an isoperimetric curve is a polar graph in Euclidean coordinates and hence in the original coordinates. Note that all constant generalized curvature curves are convex in Euclidean coordinates. \( \square \)

**Definition 4.4.** An unduloid is a nonconstant positive polar graph with constant generalized curvature.

Here we state another result from Dahlberg et al. that will be useful in classifying the potential minimizers for the \( \theta_0 \)-sector with density \( r^p \).

**Proposition 4.5.** [DDNT, Prop. 2.11] In a planar domain with density \( r^p \), \( p > 0 \), if a constant generalized curvature closed curve passes through the origin, it must be a circle.

**Lemma 4.6.** In the \( \theta_0 \)-sector with density \( r^p \), isoperimetric curves are either circles about the origin, semicircles through the origin, or unduloids.

**Proof.** By Proposition 4.3, minimizers must be polar graphs with constant generalized curvature bounding regions that contain the origin. Therefore it can either be constant, a circle, or nonconstant, an unduloid. If the curve goes through the origin, it must be part of a circle through the origin (Prop. 4.5). However, to meet regularity conditions, the curve must be an integer number of semicircles. Since one semicircle is better than \( n \) semicircles, the minimizer will be a single semicircle through the origin. \( \square \)

**Proposition 4.7.** [DDNT, Lem. 3.7] In the plane with density \( r^p \), \( p > 0 \), the least-perimeter ‘isoperimetric’ function \( I(A) \) satisfies

\[
I(A) = cA^{p+1/(p+2)}.
\]

**Remark.** While this result is stated in the plane, it also holds in the sector.

**Theorem 4.8.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > 0 \), circular arcs are isoperimetric for \( \theta_0 = \pi/(p+1) \).

**Proof.** Transition to Euclidean coordinates, where \( \theta_0 \)-sector gets mapped to the half plane. Assume some \( r(\theta) \) other than the circle is isoperimetric. By Proposition 3.5, \( r(0) \neq r(\pi) \), and \( r'(\theta) = 0 \) at 0 and \( \pi \), and nowhere else. Reflect \( r \) over the x-axis, obtaining a closed curve. By the four-vertex theorem [O], this curve has at least four extrema of classical curvature. Generalized curvature in Euclidean coordinates is Riemannian curvature divided by the area density [CJQW, Def. 3.1]. That is:

\[
\kappa_\phi = cr^{p/p+1}\kappa
\]

for some \( c > 0 \). At an extremum of Riemannian curvature we see

\[
0 = \frac{d}{d\theta}\kappa_\phi = \kappa'cr^{p/p+1} + c'r^{-1/p+1}\kappa = c'r^{-1/p+1}\kappa,
\]

for some \( c > 0 \). At an extremum of Riemannian curvature we see

\[
0 = \frac{d}{d\theta}\kappa_\phi = \kappa'cr^{p/p+1} + c'r^{-1/p+1}\kappa = c'r^{-1/p+1}\kappa.
\]
which implies either \( \dot{r} = 0 \) or \( \kappa = 0 \). However, if \( \kappa = 0 \), the curve is the geodesic, which is a straight line in Euclidean coordinates, which cannot be isoperimetric. Therefore \( \dot{r} = 0 \), meaning \( r \) must have a critical point other than 0 and \( \pi \), so it cannot be isoperimetric.

\( \square \)

**Remark.** After finding this proof and examining the isoperimetric inequality in Proposition 4.26, we came across a more geometric proof. We show that in the Euclidean \( \theta_0 \)-sector with area density \( cr^{-p/p+1} \), circles about the origin are isoperimetric for \( \theta_0 = \pi \). When \( p = 0 \), a semicircle about the origin is a minimizer. Now, for any \( p > 0 \), suppose some region \( R \) is a minimizer. Take a semicircle about the origin bounding the same Euclidean area; clearly it will have less perimeter. However, it also has more weighted area, because we have moved sections of \( R \) that were further away from the origin towards the origin. Since the area density is strictly decreasing in \( r \), we must have increased area. Therefore the circle about the origin is the minimizer for the Euclidean \( \theta_0 \)-sector with area density \( cr^{-p/p+1} \) for \( \theta_0 = \pi \), implying circular arcs are the minimizers in the \( \pi/(p+1) \)-sector with density \( r^p \).

**Proposition 4.9.** When \( p = 1 \) circles about the origin are minimizing in the sector of \( 2 \) radians with density \( r^p \).

**Proof.** Morgan [M3, Prop. 1] proves that in cones over the square torus \( T^2 = S^1(a) \times S^1(a) \) balls about the origin are minimizing as long as \( |T^2| \leq |S^2(1)| \). We note that this proof still holds for rectangular tori \( T^2 = S^1(a) \times S^1(b) \), \( a \geq b \) so long as the ratio \( a : b \) is at most \( 4 : \pi \). Taking the cone over the rectangular torus with side lengths \( 4 \) and \( \pi \), and modding out by the shorter copy of \( S^1 \) we get the cone over an angle 4 with density \( \pi r \). Since balls about the origin are minimizers in the original space, this implies that their images, circles about the origin, are minimizing in this quotient space.

\( \square \)

**Remark.** Morgan and Ritoré [MR, Rmk. 3.10] ask whether \( |M^n| \leq |S^n(1)| \) is enough to imply that balls about the origin are isoperimetric in the cone over \( M \). Trying to take the converse to the above argument we found an easy counterexample to this question. Namely taking \( M \) to be a rectangular torus of area \( 4\pi \) with one very long direction and one very short direction, we see that balls about the origin are not minimizing, as you can do better with a circle through the origin cross the short direction.

Along the same lines of Proposition 4.9 one might hope to get bounds for other values of \( p \) by examining when balls about the origin are minimizing in cones over \( S^1(\theta) \times M^p \) for \( M \) compact with a transitive isometry group and then modding out by the symmetry group of \( M \) to get the cone over \( S^1(\theta) \) with density proportional to \( r^p \). In order to see when balls about the origin are minimizing in the cone over \( T^2 \), Morgan uses the Ros product theorem with density [M2, Thm. 3.2], which requires the knowledge of the isoperimetric profile of the link (in his case \( T^2 \)). One such manifold of the form \( S^1 \times M \) for which the isoperimetric problem is solved is \( S^1 \times S^2 \) [PR, Thm. 4.3]. The three types of minimizers in \( S^1 \times S^2 \) are balls or complements of balls, tubular neighborhoods of \( S^1 \times \{\text{point}\} \), or regions bounded by two totally geodesic copies of \( S^2 \). By far the most difficult of the three cases to deal with are the balls, where unlike the other two cases we cannot explicitly compute the volume and surface area. Fixing the volume of \( S^1 \times S^2 \) as \( 2\pi^2 = |S^2| \) we can apply the same argument Morgan uses in [M3, Lemma 2] to show that if
the sectional curvature of $S^1 \times S^2$ is bounded above by 1 (by taking $S^2$ large and $S^1$ small), then balls in $S^1 \times S^2$ do worse than balls in $S^3$, as desired, but unfortunately tubular neighborhoods of $S^1$ then sometimes beat balls in $S^3$ for certain volumes, making it so we cannot apply the Ros product theorem. So without a better way to deal with balls in $S^1 \times S^2$ without such a strong assumption on the sectional curvature, this method does not work even for the $p = 2$ case. Perhaps there is some other way to prove when balls about the vertex are isoperimetric in the cone over $S^1 \times S^2$ without the Ros product theorem.

**Proposition 4.10.** In the $\theta_0$-sector with density $r^p$, semicircles through the origin are not isoperimetric for $\theta_0 < \pi (p + 2) / (2p + 2)$.

**Proof.** In Euclidean coordinates, semicircles through the origin terminate at the angle $\theta = (\pi/2)(p + 1)$. Since the semicircle approaches this axis tangentially, for any $\theta_0 < (\pi/2)(p + 2)/(p + 1)$, there is a line normal to the boundary $\theta_0(p + 1)$ in Euclidean coordinates which intersects the semicircle at a single point $b$. Replacing the segment of the semicircle from $b$ to the origin with this line increases area while decreasing perimeter. Therefore semicircles are not minimizing. \qed

**Lemma 4.11.** If the $\theta_0$-sector with density $r^p$ has isoperimetric ratio $I_0$ and

$$I_0 < (p + 2)^{p+1} \theta_0 \left[ \left( \frac{1}{p + 2} \right) + \left( \frac{p + 1}{p + 2} \right) \theta_0 \right]^{p+2},$$

then there exists an $\epsilon > 0$ such that the isoperimetric ratio of any $(\theta_0 + t)$-sector with $0 \leq t < \epsilon$ is greater than or equal to $I_0$.

**Proof.** Consider an isoperimetric curve $\gamma$ in the $(\theta_0 + t)$-sector bounding area 1. By reflection we can assume it is nonincreasing (Cor. 3.6). We partition the $(\theta_0 + t)$-sector into a $\theta_0$-sector followed by a $t$-sector and we let $\alpha_0$ denote the area bounded by $\gamma$ in the $\theta_0$-sector and $\alpha_t$ the area bounded by $\gamma$ in the $t$-sector. Note $\alpha_0 + \alpha_t = 1$. Then we can bound the isoperimetric ratio $I_t$ of the $(\theta_0 + t)$-sector by using the isoperimetric ratios $I_0$ for the $\theta_0$-sector and $R_t$ for the $t$-sector as follows:

$$I_t^{1/(p+2)} = P(\gamma) \geq (I_0 \alpha_0^{p+1})^{1/(p+2)} + (R_t \alpha_t^{p+1})^{1/(p+2)}.$$ 

Since the radius is nonincreasing we know that the $\theta_0$-sector contains at least its angular proportion of the area and thus $\alpha_0 \geq \theta_0/(\theta_0 + t)$. We now substitute $\alpha_t = 1 - \alpha_0$ and look at the right side of the inequality as a function of $\alpha_0$:

$$f_t(\alpha_0) = (I_0 \alpha_0^{p+1})^{1/(p+2)} + (R_t (1 - \alpha_0)^{p+1})^{1/(p+2)}.$$ 

This function is concave, so it attains its minimum at an endpoint. Since $\alpha_0$ is bounded between $\theta_0/(\theta_0 + t)$ and 1, we see that $I_t^{1/(p+2)}$ is greater than or equal to the minimum of $f_t(\theta_0/(\theta_0 + t))$ and $f(1)$. Since $f(1) = I_0^{1/(p+2)}$, we want to show that there is some $\epsilon$ such that $t < \epsilon$ implies $f_t(\theta_0/(\theta_0 + t)) \geq f(1) = I_0^{1/(p+2)}$. So we define a new function

$$g(t) = f_t(\theta_0/(\theta_0 + t)) - I_0^{1/(p+2)}.$$ 

We want to show that there is some positive neighborhood of 0 where $g(t) > 0$. For $t < \pi/(p + 1)$, isoperimetric regions are circular arcs, so for $t \in (0, \delta)$, $R_t = \ldots$
\[ (p + 2)^{p+1} t \] and \( g \) is differentiable. Furthermore, \( \lim_{t \to 0} g(t) = 0 \) and so it suffices to prove that \( \lim_{t \to 0} g'(t) > 0 \). We calculate
\[
\lim_{t \to 0} g'(t) = -I_0^{1/(p+2)} \left( \frac{p+1}{p+2} \right) \theta_0^{-1} + (p + 2)^{(p+1)/(p+2)} \left[ \left( \frac{1}{p+2} \right) \theta_0^{-(p+1)/(p+2)} + \left( \frac{p+1}{p+2} \right) \theta_0^{1/(p+2)} \right]
\]
and deduce that this is greater than 0 if and only if
\[ I_0 < (p + 2)^{p+1} \theta_0 \left[ \left( \frac{1}{p+2} \right) + \theta_0 \right]^{p+2} \].

**Corollary 4.12.** If the \( \theta_0 \)-sector with density \( r^p \) has isoperimetric ratio \( I_0 \) and
\[ I_0 < (p + 2)^{p+1} \theta_0 \left[ \left( \frac{1}{p+2} \right) + \theta_0 \right]^{p+2} , \]
then there exists an \( \epsilon > 0 \) such that the isoperimetric ratio is nondecreasing on the interval \( [\theta_0, \theta_0 + \epsilon) \).

**Proof.** The result follows from the continuity of the isoperimetric ratio in \( \theta_0 \) and Lemma 4.11. \( \Box \)

**Proposition 4.13.** For fixed \( p > 0 \), the isoperimetric ratio of the \( \theta_0 \)-sectors with density \( r^p \) is a nondecreasing function of \( \theta_0 \) for \( \theta_0 > p/(p+1) \).

**Proof.** When \( \theta_0 > p/(p+1) \) we have
\[ (p + 2)^{p+1} \theta_0 < (p + 2)^{p+1} \theta_0 \left[ \left( \frac{1}{p+2} \right) + \theta_0 \right]^{p+2} . \]
Since \( (p + 2)^{p+1} \theta_0 \) is the isoperimetric ratio of the circular arc in the \( \theta_0 \)-sector we see that for these \( \theta_0 \) the isoperimetric ratios \( I_0 \) always satisfy the conditions of Corollary 4.12. So, for any \( \theta_1 > \theta_0 \) we can apply Corollary 4.12 to get some nondecreasing interval starting at \( \theta_1 \) and then extend it on the right to a maximal nondecreasing interval. Since the isoperimetric ratio is continuous in \( \theta_0 \) we can take this interval to be closed and then if it is bounded we get a contradiction by applying Corollary 4.12 to its right endpoint. So, we see that for each \( \theta_1 > \theta_0 \) the isoperimetric ratio is nondecreasing on \( [\theta_1, \infty) \) and thus we deduce that the isoperimetric ratio is nondecreasing on \( (\theta_0, \infty) \). \( \Box \)

**Corollary 4.14.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > 0 \), if the semicircle through the origin is isoperimetric, it uniquely minimizes for all \( \theta > \theta_0 \).

**Proof.** By Proposition 4.13, the isoperimetric ratio is nondecreasing for \( \theta_0 > p/(p+1) \). However, for curves that don’t terminate, Lemma 4.11 and Proposition 4.13 actually show the isoperimetric ratio is strictly increasing for \( \theta_0 > 1 \). Since the semicircle through the origin does not exist before \( \pi/2 > p/(p+1) \), if it is a minimizer for \( \theta_0 \), it minimizes uniquely for all angles greater than \( \theta_0 \). \( \Box \)

**Corollary 4.15.** The semicircle is the unique minimizer in the “half-infinite parking garage” \( \{ (\theta, r) | \theta \geq 0, r > 0 \} \) with density \( r^p \).
Proof. Suppose \( \gamma \) is a minimizer in the half-infinite parking garage. Then for any \( \theta_0 \geq \pi \) the restriction of \( \gamma \) to the \( \theta_0 \) sector has isoperimetric ratio greater than that of the semicircle. Since \( \lim_{\theta_0 \to \infty} P(\gamma|\theta_0) = P(\gamma) \) and \( \lim_{\theta_0 \to \infty} A(\gamma|\theta_0) = A(\gamma) \) the limit of the isoperimetric ratios of \( \gamma|\theta_0 \) is the isoperimetric ratio of \( \gamma \) and so we see it is also greater than that of the semicircle. Since the semicircle exists in the half-infinite parking garage we are done.

Remark. Studying the isoperimetric ratio turns out to be an extremely useful tool in determining the behavior of semicircles for \( \theta_0 > \pi \). However, it is not the only such tool. Here we give an entirely different proof that semicircles minimize for all \( n\pi \)-sectors.

**Proposition 4.16.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > 0 \), even when allowing multiplicity greater than one, isoperimetric regions will not have multiplicity greater than one.

**Proof.** A region \( R \) with multiplicity may be decomposed as a sum of nested regions \( R_j \) with perimeter and area [M1, Fig. 10.1.1]:

\[
P(R) = \sum P(R_j), \quad A(R) = \sum A(R_j).
\]

Let \( R' \) be an isoperimetric region of multiplicity one and the same area as \( R \). By scaling, for each region \( P_j \geq cA_j^{(p+1)/(p+2)} \), where \( c = P_{c}^p + 2/A_{c}^{p+1} \) (Prop. 4.7). By concavity

\[
P(R) = \sum P(R_j) \geq c \sum \left( A(R_j)^{p+2} \right)^{\frac{p+2}{p+1}} \geq c \left( \sum A(R_j)^{p+2} \right)^{\frac{p+1}{p+2}} = P(R'),
\]

with equality only if \( R \) has multiplicity one. Therefore no isoperimetric region can have multiplicity greater than one.

Remark. For \( p < -2 \) the isoperimetric function \( I(A) = cA^{(p+1)/(p+2)} \) is now convex. Therefore, regions with multiplicity greater than one can do arbitrarily better than regions with multiplicity one.

**Corollary 4.17.** In the \( n\pi \)-sector (\( n \in \mathbb{Z} \)), semicircles through the origin are uniquely isoperimetric.

**Proof.** Assume there is an isoperimetric curve \( r(\theta) \) bounding a region \( R \) which is not the semicircle. Consider \( R \) as a region with multiplicity in the half plane by taking \( r(\theta) \to r(\theta \mod \pi) \). Since semicircles are the minimizers in the half plane, by Proposition 4.16, \( r \) cannot be isoperimetric. This implies that \( r \) could not have been isoperimetric in the \( n\pi \)-sector.

**Proposition 4.18.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > -1 \), circles about the origin have nonnegative second variation if and only if \( \theta_0 \leq \pi/\sqrt{p+1} \). When \( p \leq -1 \), circles about the origin always have nonnegative second variation.

**Proof.** By Proposition 3.2 we think of the \( \theta_0 \)-sector as the cone of angle \( 2\theta_0 \). A circle of radius \( r \) in the \( \theta_0 \)-sector corresponds with a circle about the axis with radius \( r\theta_0/\pi \), giving the cone the metric \( ds^2 = dr^2 + (r\theta_0/\pi)^2 d\theta^2 \). For a smooth Riemannian disk of revolution with metric \( ds^2 = dr^2 + f(r)^2 d\theta^2 \) and density \( e^{\psi(r)} \), circles of revolution at distance \( r \) have nonnegative second variation if and only if
$Q(r) = f'(r)^2 - f(r) f''(r) - f(r)^2 \psi''(r) \leq 1$ [EMMP, Thm. 6.3]. This corresponds to $(\theta_0/\pi)^2 + p(\theta_0/\pi)^2 \leq 1$, which, for $p \leq -1$, always holds. When $p > -1$, the condition becomes $\theta_0 \leq \pi/\sqrt{p+1}$, as desired. 

The following theorem is the main result of this paper.

**Theorem 4.19.** Given $p > 0$, there exist $0 < \theta_1 < \theta_2 < \infty$ such that in the $\theta_0$-sector with density $r^p$, isoperimetric curves are (see Fig. 1.1):
1. for $0 < \theta_0 < \theta_1$, circular arcs about the origin,
2. for $\theta_1 < \theta_0 < \theta_2$, unduloids,
3. for $\theta_2 < \theta_0 < \infty$, semicircles through the origin.

Moreover,
$$\pi/(p+1) < \theta_1 \leq \pi/\sqrt{p+1},$$
$$\pi(p+2)/(2p+2) \leq \theta_2 \leq \pi.$$

When $p = 1$, $\theta_1 \geq 2 > \pi/2 \approx 1.57$.

**Proof.** By Lemma 4.6, minimizers must be circles, unduloids, or semicircles. As $\theta$ increases, if the circle is not minimizing, it remains not minimizing (Prop. 4.2). If the semicircle is minimizing, it remains uniquely minimizing (Cor. 4.14). Therefore transitional angles $0 \leq \theta_1 \leq \theta_2 < \infty$ exist. Strict inequalities are trivial consequences of the following inequalities.

To prove $\theta_1 > \pi/(p+1)$, note that otherwise, because circular arcs are the unique minimizers for $\theta_0 = \pi/(p+1)$ (Thm. 4.8), there would have to be a family of other minimizers (unduloids, because certainly not semicircles) approaching the circle. This family approaches smoothly because by the theory of differential equations constant generalized curvature curves depend smoothly on their parameters. This would imply that the circle has nonpositive second variation, contradicting Proposition 4.18.

To prove $\theta_1 \leq \pi/\sqrt{p+1}$, recall that circular arcs do not have nonnegative second variation for $\theta_0 > \pi/\sqrt{p+1}$ (Prop. 4.18). To prove $\theta_2 \geq \pi(p+2)/(2p+2)$, just recall that semicircles cannot minimize for $\theta_0 < \pi(p+2)/(2p+2)$ (Prop. 4.10). To prove $\theta_2 \leq \pi$, recall that semicircles minimize for $\theta_0 = \pi$ (Prop. 4.1). Finally, when $p = 1$, circles minimize for $\theta_0 = 2$ (Prop. 4.9). 

**Remark.** For $p < -2$, circles about the origin minimize for all sectors. The proof in Euclidean coordinates given by Carroll et al. [CJQW, Prop. 4.3] generalizes immediately from the plane to the sector. This proof can also be translated into the original coordinates.

We conjecture that the circle is minimizing as long as it has nonnegative second variation, and that the semicircle is minimizing for all angles greater than $\pi(p+2)/(2p+2)$.

**Conjecture 4.20.** In Theorem 4.19, the transitional angles $\theta_1$, $\theta_2$ are given by $\theta_1 = \pi/\sqrt{p+1}$ and $\theta_2 = \pi(p+2)/(2p+2)$.

**Remark.** This conjecture is supported by numeric evidence as in Figure 1.2. Our Maple program ([DHHT, Sect. 10]) predicts the circular arc stops minimizing at exactly $\pi/\sqrt{p+1}$, and predicts the semicircle begins to minimize very close to $\pi(p+2)/(2p+2)$.
Remark. Our minimizing unduloids give explicit examples of the abstract existence result of Rosales et al. [RCBM, Cor. 3.13] of isoperimetric regions not bounded by lines or circular arcs.

One potential avenue for proving this conjecture is discussed in Proposition 5.2. We also believe the transition between the circle and the semicircle is parametrized smoothly by curvature, which is discussed in Section 5.

Corollary 4.22 gives an example of the applicability of Theorem 4.19, as suggested to us by Antonio Cañete, extending to certain polygons with density an isoperimetric theorem for polytopes of Morgan [M2, Thm. 3.8]. First we need a small lemma:

**Lemma 4.21.** Consider a polygon in the Euclidean plane. Then there exists a $c > 0$ such that any region in the polygon bounding area $A$ less than half the area of the polygon with perimeter $P$ satisfies the following inequality:

$$\frac{P^2}{A} \geq c.$$

**Proof.** This inequality is known for the circle. Mapping the polygon to a circle such that the factor by which distance is stretched is bounded above and below, we see it holds in a polygon, although the constant $c$ depends on the polygon. □

**Corollary 4.22.** Consider a polygon with a vertex of angle $\theta_0$ located at the origin in the plane with density $r^p$, $p > 0$. For sufficiently small area, the isoperimetric curve bounding that area will be the same as in the $\theta_0$-sector with density $r^p$.

**Proof.** Let $r_0$ be small enough so that $B(0, r_0)$ intersects only one vertex (the origin) and two edges. Consider areas small enough that any region of that area has less than half the Euclidean area of the polygon, and so that an isoperimetric region of that area has perimeter less than $(r_0/2)^{p+1}$, so that any curve from the circle of radius $r_0/2$ to the circle of radius $r_0$ has more weighted perimeter. An isoperimetric region inside the circle of radius $r_0$ satisfies

$$P = cA^{(p+1)/(p+2)}$$

(Prop. 4.7). We claim that any region outside the circle of radius $r_0/2$ satisfies $P \geq c'A^{1/2}$.

Let $R$ be a (possibly disconnected) isoperimetric region in the polygon with perimeter $P$ and area $A$. We note that every component of $R$ is either inside $B(0, r_0)$ or outside $B(0, r_0/2)$. Suppose there is a component outside $B(0, r_0/2)$ with area $A_1$ and perimeter $P_1$. By Lemma 4.21 and because $r$ is bounded above and below this component satisfies the following inequality:

$$P_1 \geq c''A_1^{1/2}.$$

Since for sufficiently small $A_1$

$$cA_1^{(p+1)/(p+2)} < c''A_1^{1/2},$$

we see the best curve inside $B(0, r)$ bounding area $A_1$ will have less perimeter than the best curve outside $B(0, r/2)$ bounding area $A_1$. This implies $R$ could have the same area with less perimeter as a region with multiplicity in $B(0, r)$. This is the same as a region with multiplicity in the sector, which by Proposition 4.16 cannot be isoperimetric. Therefore for sufficiently small area an isoperimetric curve must
lie inside $B(0, r)$, meaning the isoperimetric curve will be the same as that in the sector.

\[ \square \]

**Proposition 4.23.** For any $n \in \mathbb{R} \backslash \{0\}$ the $\theta_0$-sector with perimeter density $r^p$ and area density $r^q$ is equivalent to the $[n] \theta_0$-sector with perimeter density $r^{(p+1)/n - 1}$ and area density $r^{(q+2)/n - 2}$.

**Proof.** Make the coordinate change $w = z^n / n$. \[ \square \]

Using Proposition 4.23 and our results in the sector (Thm. 4.19) we obtain the following proposition:

**Proposition 4.24.** In the plane with perimeter density $r^k$, $k > -1$, and area density $r^m$ the following are isoperimetric curves:

1. For $m \in (-\infty, -2] \cup (2k, \infty)$ there are none.
2. For $k \in (-1, 0)$ and $m \in (-2, 2k)$ the circle about the origin.
3. For $k \in [0, \infty)$ and $m \in (-2, k-1]$ the circle about the origin.
4. For $k \in [0, \infty)$ and $m \in [k, 2k]$ pinched circles through the origin.

We also obtain a conjecture on the undecided area density range between $k - 1$ and $k$ based on our conjecture on sectors (Conj. 4.20).

**Conjecture 4.25.** In the plane with perimeter density $r^k$, $k > -1$, and area density $r^m$ the following are isoperimetric curves:

1. For $k \in [0, \infty)$ and $m \in (k-1, k-1 + \frac{1}{k+1}]$ the circle about the origin.
2. For $k \in [0, \infty)$ and $m \in (k-1 + \frac{1}{k+1}, k-1 + \frac{k+1}{2k+1})$ unduloids.
3. For $k \in [0, \infty)$ and $m \in [k-1 + \frac{k+1}{2k+1}, k]$ pinched circles through the origin.

**Remark.** Along the lines of Conjecture 4.25, the circular arc being the minimizer up to the $\pi/(p+1)$ sector is equivalent to the circle being the minimizer in the Euclidean plane with any perimeter density $r^p$, $p \in [0, 1]$ and area density 1. As $p/2 + 1$ is the tangent line to $\sqrt{p+1}$ at 0, this is the best possible bound we could obtain which is linear in the denominator.

We now consider a more analytic formulation of the isoperimetric problem in the $\theta_0$-sector with density $r^p$, and give an integral inequality that is equivalent to proving the conjectured angle of $\theta_1$. The following integral inequality follows directly from the definitions of weighted area and perimeter.

**Proposition 4.26.** In the $\theta_0$-sector with density $r^p$, $p > 0$, circles about the origin are isoperimetric if and only if the inequality

\[
\int_0^1 r^{p+2} d\alpha \leq \int_0^1 \sqrt{r^2 + \frac{r^2}{(p+1)\theta_0}^2} d\alpha
\]

holds for all $C^1$ functions $r(\alpha)$.

**Remark.** This gives a nice analytic proof that circles about the origin minimize for $\theta_0 = \pi/(p+1)$; letting $\theta_0 = \pi/(p+1)$, we have

\[
\int_0^1 r^{p+2} d\alpha \leq \int_0^1 \sqrt{r^2 + \frac{r^2}{\pi^2}} d\alpha.
\]
Figure 5.1. Types of constant generalized curvature curves: a) for $0 < \lambda < p + 2$ nonconstant periodic polar graphs (unduloids), b) for $\lambda < 0$ or $\lambda > p + 2$ periodic nodoids, c) for $\lambda = p + 2$, a circle through the origin, for $\lambda = p + 1$ a circle about the origin, for $\lambda = 0$ a curve asymptotically approaching the radial lines $\theta = \pm \pi/(2p + 2)$.

When $p = 0$, this corresponds to the isoperimetric inequality in the half-plane with density 1. As pointed out by Leonard Schulman of CalTech, the left hand side is nonincreasing as a function of $p$, meaning the inequality holds for all $p > 0$.

**Corollary 4.27.** In the $\theta_0$-sector with density $r^p$, $p > 0$, circles about the origin are isoperimetric for $\theta_0 = \pi/\sqrt{p + 1}$ if and only if the inequality

$$\left[ \int_0^1 r^q \, d\alpha \right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q - 1) \frac{\dot{r}^2}{\pi^2}} \, d\alpha$$

holds for all $C^1$ functions $r(\alpha)$ for $1 < q \leq 2$.

**Proof.** In the inequality from Corollary 4.26, let $q = (p + 2)/(p + 1)$, and let $\theta_0 = \pi/\sqrt{p + 1}$. \hfill $\square$

**Remark.** We wonder if an interpolation argument might work here. When $q = 1$, the result holds trivially (equality for all functions $r$), and when $q = 2$, the inequality follows from the isoperimetric inequality in the half plane.

### 5. Constant Generalized Curvature Curves

We look at constant generalized curvature curves in greater depth. Theorem 5.1 classifies constant generalized curvature curves. Proposition 5.2 proves that if the half period of constant generalized curvature curves is bounded above by $\pi(p + 2)/(2p + 2)$ and below by $\pi/\sqrt{p + 1}$, then Conjecture 4.20 holds. Our major tools for studying constant generalized curvature curves are the second order constant generalized curvature equation and its first integral.

**Theorem 5.1.** In the plane with density $r^p$, a curve with constant generalized curvature $\lambda$ normal at $(1, 0)$ is (see Fig. 5.1):

1. for $\lambda \in (0, p + 2) - \{p + 1\}$, a periodic unduloid,
2. for $\lambda \notin [0, p + 2]$, a periodic nodoid,
3. for $\lambda = p + 2$, a circle through the origin,
(4) for $\lambda = p + 1$, a circle about the origin,
(5) for $\lambda = 0$, the geodesic

$$r(\theta) = (\sec((p + 1)\theta))^{1/(p+1)}$$

which asymptotically approaches the radial line $\theta = \pm \pi/(2p + 2)$.

**Proof.** These results follow from the differential equations for constant generalized curvature curves. Two points deserve special mention. First, reflection and scaling provides a correspondence between curves outside the circle and curves inside the circle. Second, to prove a curve $\gamma$ which is not the circle, semicircle or geodesic is periodic, it suffices to show $\gamma$ has a critical point after $(1,0)$. It follows directly from the differential equation that there is a radius at which $\gamma$ would attain a critical point. To prove $\gamma$ actually attains this radius, we note that otherwise $\gamma$ would have to asymptotically spiral to this radius. This implies $\gamma$ has the same generalized curvature as a circle of this radius, which the equations confirm cannot happen. \hfill $\Box$

Isoperimetric curves in the $\theta_0$-sector are polar graphs with constant generalized curvature that intersect the boundary perpendicularly. Given these restrictions and the classification of Theorem 5.1, we can give a sufficient condition for Conjecture 4.20 on the value of the transitional angles.

**Proposition 5.2.** In the plane with density $r^p$, assume the half period of constant generalized curvature curves normal at $(1,0)$ with generalized curvature $0 < \lambda < p + 1$ is bounded below by $\pi/\sqrt{p + 1}$ and above by $\pi(p + 2)/(2p + 2)$. Then the conclusions of Conjecture 4.20 hold.

**Proof.** Given the correspondence mentioned in the proof of Theorem 5.1, it suffices to only check curves with $0 < \lambda < p + 1$. By Proposition 4.6, minimizers are either circles about the origin, semicircles through the origin, or unduloids. An unduloid is only in equilibrium when the sector angle is equal to its half period. If there are no unduloids with half period less than $\pi/\sqrt{p + 1}$ or greater than $\pi(p + 2)/(2p + 2)$, the only possible minimizers outside of that range are the circle and the semicircle. The proposition follows by the bounds given in Theorem 4.19. \hfill $\Box$

**Remark 5.3.** We now discuss some results on the periods of constant generalized curvature curves, and provide numeric evidence. Using the second order constant generalized curvature equation, we proved that curves with generalized curvature near that of the circle have periods near $\pi/\sqrt{p + 1}$ [DHHT, Sect. 6]. Similarly, using the first integral of the constant generalized curvature equation for $dr/d\theta$, we see that the half period of a curve with constant generalized curvature is given by

$$T = \int_1^{r_1} \frac{dr}{r^{p+2} \left( \frac{1-r^p}{1-r^{p+1}} \right) - 1}$$

where $r_1$ is the curve’s maximum radius [DHHT, Sect. 6]. Using this formula, we generated numeric evidence for the bounds given in Proposition 5.2, as seen in Figure 5.2. This plot is given for $p = 2$, and is representative of all $p$. Moreover, the integral appears to be monotonic in $r_1$ which would imply that every unduloid with $0 < \lambda < p + 2$ minimizes for exactly one $\theta_0$-sector. Francisco López has suggested studying the integral above with the techniques of complex analysis.
6. The Isoperimetric Problem in Sectors with Disk Density

In this section we classify the isoperimetric curves in the $\theta_0$-sector with density $1$ outside the unit disk $D$ centered at the origin and $a > 1$ inside $D$. Cañete et al. [CMV, Sect. 3.3] consider this problem in the plane, which is equivalent to the $\pi$-sector. Proposition 6.2 gives the potential minimizing candidates. Theorems 6.3, 6.4, 6.5 classify the isoperimetric curves for every area and sector angle.

**Definition 6.1.** A *bite* is an arc of $\partial D$ and another internal arc (inside $D$), the angle between them equal to $\arccos(1/a)$ (see Fig. 6.1(c)).

**Proposition 6.2.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and $1$ outside, for area $A > 0$, an isoperimetric set is one of the following (see Fig. 6.1):

1. an arc about the origin inside or outside $D$;
2. an annulus inside $D$ with $\partial D$ as a boundary;
3. a bite;
4. a semicircle on the edge disjoint from the interior of $D$;
5. a semicircle centered on the $x$-axis enclosing $D$ for $\theta_0 = \pi$.

**Proof.** Any component of a minimizer has to meet the edge normally by Proposition 3.4, since if not rotation about the origin brings it into contact with the edge or another component, contradicting regularity (Prop. 2.5). Since each part of the boundary has to have constant generalized curvature it must be made up of circular arcs. We can discard the possibility of combinations of circular arcs with the same density since one circular arc is better than $n$ circular arcs. Therefore, there are five possible cases:

1. A circular arc from one boundary edge to itself (including possibly the origin).
Figure 6.1. Isoperimetric sets in sectors with disk density: (a) an arc about the origin inside or outside $D$, (b) an annulus inside $D$, (c) a bite, (d) a semicircle on the edge disjoint from the interior of $D$, (e) a semicircle centered on the $x$-axis enclosing $D$ for $\theta_0 = \pi$.

There are three possibilities according to whether the semicircle has 0, 1, or 2 endpoints inside the interior of $D$. A semicircle with two endpoints inside $D$ has an isoperimetric ratio of $2\pi a$. For a semicircle with one endpoint inside $D$ (Fig. 6.2), by Proposition 2.5 Snell’s Law holds. Then the only possible curve that intersects the boundary normally would also have to intersect $D$ normally. Its perimeter and area satisfy:

$$P = r(\pi - \beta + a\beta),$$

$$A = \frac{r^2}{2} (\pi - \beta + a\beta),$$

Figure 6.2. A semicircle meeting $\partial D$ perpendicularly has more perimeter than a semicircle disjoint from the interior of $D$ (Fig. 6.1d).
Figure 6.3. A semicircle tangent to $\partial D$ together with the rest of $\partial D$ is never isoperimetric.

Therefore a semicircle (d) outside $D$ with isoperimetric ratio $2\pi$ is the only possibility.

(2) A circular arc from one boundary edge to another.
An arc (a) or annulus (b) about the origin are the only possibilities for any $\theta_0 < \pi$ or $A < a\theta_0/2$. At $\theta_0 = \pi$ and area $A > a\theta_0/2$ a semicircle (e) centered on the x-axis enclosing $D$ is equivalent to a semicircle centered at the origin. For $\theta_0 > \pi$ and $A > a\theta_0/2$, a semicircle tangent to $\partial D$ together with the rest of $\partial D$ (Fig. 6.3) is in equilibrium, but we will show that it is not isoperimetric. Its perimeter and area satisfy:

$$P = \pi R + (\theta_0 - \pi), \quad A = \frac{\pi}{2} (R^2 - 1) + \frac{\theta_0 a}{2}.$$ 

Comparing it with an arc (a) about the origin we see that it is not isoperimetric.

(3) Two circular arcs (c) meeting along $\partial D$ according to Snell’s Law (Prop. 2.5).

(4) Three or more circular arcs meeting along $\partial D$.

By Cañete et al. [CMV, Prop. 3.19] this is never isoperimetric.

(5) Infinitely many circular arcs meeting along $\partial D$.

By Cañete et al. [CMV, Prop. 3.19] this is never isoperimetric.

\[ \square \]

**Theorem 6.3.** For some $\theta_2 < \pi$, in the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for $\theta_0 \leq \pi$, there exists $0 < A_0 < A_1 < a\theta_0/2$, such that an isoperimetric curve for area $A$ is (see Fig. 6.1):

(1) if $0 < A < A_0$, an arc about the origin if $\theta_0 < \pi/a$, semicircles on the edge disjoint from the interior of $D$ if $\theta_0 > \pi/a$, and both if $\theta_0 = \pi/a$;

(2) if $A = A_0$, both type (1) and (3);
(3) if $A_0 \leq A < A_1$, a bite, if $A_1 < A < a\theta_0/2$ an annulus inside $D$, and if $A = A_1$ both; if $\theta_0 > \theta_2$, $A_1 = a\theta_0/2$ and the annulus is never isoperimetric;
(4) if $A \geq a\theta_0/2$, an arc about the origin; at $\theta_0 = \pi$ any semicircle centered on the $x$-axis enclosing $D$.

**Theorem 6.4.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for $\pi < \theta_0 \leq a\pi$, there exists $A_0$, $A_1$, such that an isoperimetric curve for area $A$ is (see Fig. 6.1):

1. if $0 < A < A_0$, a semicircle on the edge disjoint from the interior of $D$;
2. if $A = A_0$, both type (1) and (3);
3. if $A_0 < A < a\theta_0/2$, a bite;
4. if $a\theta_0/2 \leq A < \theta_2^0(a - 1)/2(\theta_0 - \pi)$, an arc about the origin;
5. if $A = \theta_2^0(a - 1)/2(\theta_0 - \pi)$, both type (4) and (6);
6. if $A > \theta_2^0(a - 1)/2(\theta_0 - \pi)$, a semicircle on the edge disjoint from the interior of $D$.

**Theorem 6.5.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for $\theta_0 > a\pi$, the isoperimetric curves for area $A$ are semicircles on the edge disjoint from the interior of $D$ (Fig. 6.1e).

![Figure 6.4](image.png)

**Figure 6.4.** For $\theta_0 < f$ an arc about the origin is isoperimetric, while for $\theta_0 > f$ a semicircle in the edge is isoperimetric. For area infinitesimally less than $a\theta_0/2$, for $\theta_0 < g$ an annulus inside $D$ is isoperimetric, while for $\theta_0 > g$ a bite is isoperimetric.

**Remark.** Given Proposition 6.2, most of the proofs of Theorems 6.3, 6.4, 6.5 are by simple perimeter comparison. Figure 6.4 shows numerically the transition between different minimizers. For more details see [DHHT, Sect. 7]. The hardest comparison is handled by the following sample lemma.

**Lemma 6.6.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for a fixed angle and area $A < a\theta_0/2$, once a bite encloses more area with less perimeter than an annulus inside $D$, it always does.

**Proof.** Increasing the difference $a\theta_0/2 - A$, it will be sufficient to prove that a bite is better than a scaling of a smaller bite because an annulus changes by scaling. Therefore, there will be at most one transition.
Figure 6.5. A bite is better than a scaling of a smaller bite, and hence once better than an annulus, it is always better.

To prove that a bite does better than a scaling of a smaller bite we show that it has less perimeter and more area. Take a smaller bite and scale it by $1 + \epsilon$ (see Fig. 6.5) and eliminate the perimeter outside $D$. Another bite is created with perimeter $P_a$ and area $A_a$:

$$P_a < (1 + \epsilon)P, \quad A_a = (1 + \epsilon)^2 (A_1 + A_2) - A_3 > (1 + \epsilon)^2 A_1,$$

Thus it is better than scaling a smaller bite. 

7. Symmetrization

Proposition 7.3 extends a symmetrization theorem of Ros [R1, Sect. 3.2] to warped products with a product density as asserted by Morgan [M2, Thm. 3.2], and is general enough to include spherical symmetrization as well as Steiner and Schwarz symmetrization. Proposition 7.5 analyzes the case of equality in Steiner symmetrization after Rosales et al. [RCBM, Thm. 5.2]. Proposition 7.6 extends symmetrization to Riemannian fiber bundles with equidistant Euclidean fibers. Some simple examples are lens spaces (fibered by circles) as envisioned by Antonio Ros [R2, R1, Thm. 2.11], similar Hopf circle fibrations of $S^{2n+1}$ over $\mathbb{CP}^n$, the Hopf fibration of $S^7$ by great $S^3$s and of $S^{15}$ by great $S^7$s. We were not, however, able to complete the proof by symmetrization envisioned by Vincent Bayle (private communication) of the conjecture that in $\mathbb{R}^n$ with a smooth, radial, log-convex density, balls about the origin are isoperimetric [RCBM, Conj. 3.12]. We thank Bayle and Ros for helpful conversations. Standard references on symmetrization are provided by Burago and Zalgaller [BZ, Sect. 9.2] and Chavel [C, Sect. 6]. Gromov [G, Sect. 9.4] provides some sweeping remarks and conditions for symmetrization, including fiber bundles.

Definition 7.1. The Minkowski Perimeter of a region $R$ is

$$\lim_{\delta r \to 0} \inf \frac{\delta V}{\delta r}$$

for $r$ enlargements. The limit exists and agrees with the usual definition of perimeter as long as the boundary of $R$ is rectifiable and the metric and density are continuous (see [F, Thm 3.2.39]).
Lemma 7.2. (Morgan [M4, Lem. 2.4]) Let f, h be real-valued functions on [a, b]. Suppose that f is upper semicontinuous from the left and that h is C^1. Suppose that for a ≤ r < b the lower right derivative of f satisfies

\[ f'(r) = \liminf_{\Delta r \to 0} \frac{f(r + \Delta r) - f(r)}{\Delta r} \geq h'(r). \]

Then f(b) − f(a) ≥ h(b) − h(a).

Proposition 7.3. Symmetrization for warped products. Let B be a Riemannian manifold. Consider a warped product \( B \times \mathbb{R}^n \) with metric \( ds^2 = db^2 + g(b)^2 dt^2 \), with continuous product density \( \varphi(b) \cdot \psi(t) \). Let \( R \) be a region of finite (weighted) perimeter. Suppose that in each fiber \( \{b\} \times \mathbb{R}^n \) balls about the origin are isoperimetric. Then the Schwarz symmetrization \( \text{sym}(R) \) has the same volume and no greater perimeter than \( R \).

Remark. In the statement and proof \( \mathbb{R}^n \) may be replaced by \( S^n \); also balls about the origin may be replaced by half planes \( \{x_n \leq c\} \) (for \( \mathbb{R}^{n-1} \times \mathbb{R}^+ \) as well as \( \mathbb{R}^n \)) when these have finite weighted volume.

Proof. The preservation of volume is just Fubini’s theorem.

For small \( r \), denote \( r \)-enlargements in \( B \times R \) by a superscript \( r \) and \( r \)-enlargements in fibers by a subscript \( r \). Consider a slice \( \{b_0\} \times C = R(b_0) \) of \( R \) and a ball about the origin \( \{b_0\} \times D \) in the same fiber of the same weighted volume. For general \( b \), consider slices \( \{(b_0) \times C\}^r \) of \( \{(b_0) \times C\} \) and similarly slices \( \{(b_0) \times D\}^r \) of \( \{(b_0) \times D\} \). We see that \( \{(b_0) \times C\}^r \) and \( \{(b_0) \times D\}^r \) have the same \( r \) because \( g \) is invariant under vertical translation. Because the fiber density \( \psi(t) \) is independent of \( b \), \( \{b\} \times C \) and \( \{b\} \times D \) have the same weighted volume. Since every \( \{b\} \times (D_r) \) is isoperimetric, by Lemma 7.2,

\[ |\{b\} \times (D_r)| \leq |\{b\} \times (C_r)|, \]

and hence

\[ \{(b_0) \times D\}^r \subseteq \text{sym}(\{(b_0) \times C\}^r). \]

Since this holds for all \( b \),

\[ \{(b_0) \times D\}^r \subseteq \text{sym}(\{(b_0) \times C\}^r). \]

Since this holds for all \( b_0 \),

\[ (\text{sym}(R))^r = \bigcup_{b_0 \in R} \{(b_0) \times D\}^r \subseteq \text{sym}(\{(b_0) \times C\}^r) \subseteq \text{sym}(R^r). \]

Consequently,

\[ |(\text{sym}(R))^r| \leq |R^r| \]

and \( \text{sym}(R) \) has no more perimeter than \( R \), as desired. \( \square \)

Remark. If \( \psi(t) \) is a radial function, then Schwarz symmetrization may also be deduced from Steiner symmetrization by starting with Hsiang SO(\( n-1 \)) symmetrization in \( \mathbb{R}^n \) and modding out by SO(\( n-1 \)) to reduce to Steiner symmetrization.

After Rosales et al. [RCBM, Thm. 5.2] we provide a uniqueness result for the case where the fibers are copies of \( \mathbb{R} \), which is Steiner symmetrization. First we
state a lemma of theirs. For a complete analysis of uniqueness without density or warping see Chlebík et al. [CCF].

**Lemma 7.4.** [RCBM, Lem. 5.3] Suppose that we have finitely many nonnegative real numbers with \( \sum_j \alpha_j a_j \geq 2\alpha a \) and \( \sum_j \alpha_j \geq 2\alpha \). Then the following inequality holds

\[
\sum_j \alpha_j \sqrt{1 + a_j^2} \geq 2\alpha \sqrt{1 + a^2},
\]

with equality if and only if \( a_j = a \) for every \( j \) and \( \sum_j \alpha_j = 2\alpha \).

**Proposition 7.5. Uniqueness.** Let \( \psi \) be a smooth density on \( \mathbb{R} \) such that centered intervals are uniquely isoperimetric for every prescribed volume. Let \( B \) be a smooth \( n \)-dimensional Riemannian manifold with density \( \varphi \), and consider the warped product \( B \times_{g} \mathbb{R} \) with metric \( ds^2 = db^2 + g(b)^2 dt^2 \) and product density \( \varphi(b) \cdot \psi(t) \). Let \( R \) be an measurable set in \( B \times_{g} \mathbb{R} \) such that almost every fiber intersects its topological boundary \( \partial R \) transversely where \( \partial R \) is locally an even number (possibly zero) of smooth graphs over \( B \) with nonvertical tangent planes, and let \( R' \) denote its Steiner symmetrization. Suppose that almost every fiber intersects \( \partial R' \) where it is smooth with a nonvertical tangent plane or not at all, and that the fibers that don't intersect \( \partial R' \) where it has a nonvertical tangent plane do not contribute anything to the perimeter. Suppose \( R \) and \( R' \) have the same perimeter. Then \( R = R' \) up to a set of measure zero.

**Proof.** Let \( D \) be the image of projection of \( R \) to \( B \). Let \( A \subseteq D \) be the set of points \( p \) in \( D \) for which \( \partial R \) and \( \partial R' \) both have a nonvertical tangent planes above \( p \). By the definition of Steiner symmetrization

\[
\sum_{i \text{ odd}} \int_{h_i}^{h_{i+1}} g(p) \varphi(p) \psi(x)dx = 2 \int_{0}^{h^*} g(p) \varphi(p) \psi(x)dx
\]

\[
\sum_{i \text{ odd}} \int_{h_i}^{h_{i+1}} \psi(x)dx = 2 \int_{0}^{h^*} \psi(x)dx
\]

on \( A \) where \( h^* \) is height function of \( \partial R' \) with respect to \( B \). Varying \( p \) we get that

\[
\sum_{i \text{ odd}} (\psi(h_{i+1}) \nabla h_{i+1} - \psi(h_i) \nabla h_i) = 2\psi(h^*) \nabla h^*.
\]

Hence

\[
\sum_{i \text{ odd}} \psi(h_i)|\nabla h_i| \geq 2\psi(h^*)|\nabla h^*|,
\]

\[
\sum_j \varphi(p)\psi(h_j) g(p)|\nabla h_j| \geq 2\varphi(p)\psi(h^*) g(p)|\nabla h^*|
\]

on \( A \). By the assumption that centered intervals are uniquely isoperimetric we have that

\[
\sum_j \varphi(p)\psi(h_j) \geq 2\varphi(p)\psi(h^*),
\]

with equality if and only if the corresponding slice of \( R \) is a centered interval.

By Lemma 7.4 with

\[
\alpha_j = \varphi(p)\psi(h_j(p)), \ a_j = g(p)|\nabla h_j(p)|, \ \alpha = \varphi(p)\psi(h^*(p)), \ a = g(p)|\nabla h^*(p)|
\]
we get
\[ \sum_j \varphi(p) \psi(h_j(p)) \sqrt{1 + g^2(p) \vert \nabla h_j(p) \vert^2} \geq 2 \varphi(p) \psi(h^*(p)) \sqrt{1 + g^2(p) \vert \nabla h^*(p) \vert^2}, \]
with equality if and only if the number of graphs is two (or trivially 0), \( g \vert \nabla h_1(p) \vert = g \vert \nabla h_2(p) \vert = g \vert \nabla h^*(p) \vert \) and the slice of \( R \) at \( p \) is a centered interval.

The perimeter \( P(R) \) satisfies
\[ P(R) \geq \int_A (\sum_j \varphi(p) \psi(h_j(p)) \sqrt{1 + g^2(p) \vert \nabla h_j(p) \vert^2}) \]
\[ \geq \int_A (2 \varphi(p) \psi(h^*(p)) \sqrt{1 + g^2(p) \vert \nabla h^*(p) \vert^2}) = P(R'), \]
because the fibers over \( D - A \) contribute nothing to \( P(R') \). If equality holds then \( R \) coincides with \( R' \) in almost every fiber of \( D \), i.e., \( R = R' \) up to a set of measure 0.

\[ \square \]

**Remark.** If we assume for example that \( \partial R \) and \( \partial R' \) are smooth, then it follows that \( R = R' \).

The following proposition provides for fiber bundles a similar symmetrization in a related warped product. The assumption that fibers are equidistant implies that locally parallel transport normal to one fiber yields a diffeomorphism with any nearby fiber, which we assume is an isometry or a scaling.

**Proposition 7.6. Symmetrization for fiber bundles.** Consider a Riemannian fiber bundle \( M \to B \) with equidistant Euclidean fibers \( M_b = \mathbb{R}^n \) and a smooth positive function \( g(b) \) such that parallel transport normal to the fibers from \( M_{b_1} \) to \( M_{b_2} \) scales the metric on the fibers by \( g(b_2)/g(b_1) \). Suppose that \( M \) is compact or more generally that:

1. \( B \) is compact or more generally has positive injectivity radius and
2. for some \( r_0 > 0 \), for \( r < r_0 \) the \( r \)-tube about a fiber \( M_b \) under parallel translation from that fiber has metric \( 1+o(1) \) times that of a warped product \( B(b, r) \times_g \mathbb{R}^n \), uniform in \( b \).

Let \( R \) be a region of finite perimeter. Consider the Schwarz symmetrization \( \text{sym}(R) \) in the warped product \( B \times_g \mathbb{R}^n \), which replaces the slice of \( R \) in each fiber with a ball about the origin of the same volume. Then \( \text{sym}(R) \) has the same volume and no greater perimeter than \( R \).

**Remark.** In the statement and proof the Euclidean fibers may be replaced by the other constant curvature model spaces: \( \mathbb{S}^n \) or \( \mathbb{H}^n \).

**Proof.** The preservation of volume follows from the equidistance of the fibers.

As in the proof of Proposition 7.3 denote \( r \)-enlargements in \( M \) by a superscript \( r \) and \( r \)-enlargements in fibers by a subscript \( r \). Let \( r \) be a small positive number less than both \( r_0 \) and the injectivity radius of \( B \). Consider a slice \( C = R(b_0) \) of \( R \) and a ball \( D \) of the same volume about the origin in the corresponding fiber of \( B \times_g \mathbb{R}^n \). For general \( b \), if \( C^r \) denotes the image of \( C \) in \( M_b \) under normal parallel transport and \( D' \) denotes the copy of \( D \) in \( \{ b \} \times_g \mathbb{R}^n \), \( |C^r| = |D'| \). As in the proof of Proposition 7.3, \( |D'(b)| = |D'_{r'}| \), but due to the twisting in fiber bundles, it is not necessarily true that \( |C^{r'}(b)| = |C'_{r'}| \). By the uniformity hypothesis (2), the
map by parallel transport based at $M_{b_0}$ from $B \times g \mathbb{R}^n$ to $M$ distorts the metric by $1 + o(1)$, uniform over $M$. Therefore
\[ C_r' \subseteq C^{r+o(r)}(b). \]
Now, since each $D_r'$ is isoperimetric, by Lemma 7.2,
\[ |D_r'| \leq |C_r'|. \]
And so we get
\[ D_r(b) \subseteq \text{sym}(C^{r+o(r)}(b)). \]
Since this holds for all $b$,
\[ D_r \subseteq \text{sym}(C^{r+o(r)}). \]
Since this holds for all $b_0$,
\[ (\text{sym}(R))^r = \bigcup_{b_0 \in R} D_r \subseteq \bigcup_{b_0 \in R} \text{sym}(C^{r+o(r)}) \subseteq \text{sym}(R^{r+o(r)}). \]
Consequently,
\[ |(\text{sym}(R))^r| \leq |R^{r+o(r)}|, \]
and $\text{sym}(R)$ has no more perimeter than $R$, as desired. □

Remark. In the unwarped case, including the simplest lens spaces, the $1 + o(1)$ factor in the proof is unnecessary, because distance from a point in a fixed fiber in the lens space is less than its value in the product (see [DHHT, Prop. 8.6]).

8. ISOPERIMETRIC PROBLEMS IN $\mathbb{R}^n$ WITH RADIAL DENSITY

In this section we look at the isoperimetric problem in $\mathbb{R}^n$ with a radial density. We use symmetrization (Prop. 7.3) to show that if a minimizer exists, then a minimizer of revolution exists (Lem. 8.1), thus reducing the problem to a planar problem (Lem. 8.2). We consider the specific case of $\mathbb{R}^n$ with density $r^p$ and provide a conjecture (Conj. 8.3) and a nonexistence result (Prop. 8.4).

Lemma 8.1. In $\mathbb{R}^n$ with a radial density if there exists an isoperimetric region then there exists an isoperimetric region of revolution.

Proof. $\mathbb{R}^n$ is the same as $\mathbb{R}^+ \times S^{n-1}$ with warped product metric $ds^2 = dr^2 + r^2 d\Theta^2$. By Proposition 7.3 we can replace the intersection of the isoperimetric region with each spherical shell by a polar cap, preserving area without increasing perimeter. After performing this operation the region is rotationally symmetric about an axis, because each spherical cap is. □

Remark. This is just standard spherical symmetrization (see [C, Sect 6.4]).

Lemma 8.2. The isoperimetric problem in $\mathbb{R}^n$ with density $f(r)$ is equivalent to the isoperimetric problem in the half plane $y > 0$ with density $y^{n-2} f(r)$.

Proof. Up to a constant, this is the quotient space of $\mathbb{R}^n$ with density $f(r)$ modulo rotations about an axis. By Lemma 8.1, there exists a minimizer bounded by surfaces of revolution, so a symmetric minimizer in $\mathbb{R}^n$ corresponds to a minimizer in the quotient space and vice-versa. □
Isoperimetric curves in the plane with density $r^p$ are circles about the origin when $p < -2$, do not exist when $-2 < p < 0$, and are circles through the origin when $p > 0$ ([CJQW, Props. 4.2, 4.3], [DDNT, Thm. 3.16]). We now look at the isoperimetric problem in $\mathbb{R}^n$ with density $r^p$.

**Conjecture 8.3.** In $\mathbb{R}^n$ with density $r^p$ isoperimetric regions are spheres about the origin if $p < -n$ and spheres through the origin if $p > 0$, and you can bound any volume with arbitrarily small area whenever $-n \leq p < 0$.

**Proposition 8.4.** For $-n < p < 0$ in $\mathbb{R}^n$ with density $r^p$, minimizers do not exist; you can enclose any volume with arbitrarily small perimeter.

**Proof.** Consider a sphere $S$ of radius $R \gg 0$ not containing the origin. Let $r_{\text{min}}, r_{\text{max}}$ be the minimum and maximum values of $r$ attained on $S$, note that $0 < r_{\text{min}} = r_{\text{max}} - 2R$. Let $V, P$ be the volume and perimeter respectively of $S$. We have that:

$$V > c_0 R^n \min S(r^p) = c_0 R^n r_{\text{max}}^p.$$  

Fixing $V$ we get that $r_{\text{max}} > c_1 R^q$ where $q = -n/p > 1$. This gives

$$r_{\text{min}} > c_1 R^q - 2R > c_1 \frac{1}{2} R^q$$

for sufficiently large $R$. Looking at $P$ we have

$$P < c_2 R^{n-1} \max S(r^p) = c_2 R^{n-1}r_{\text{min}}^p < c_2 R^{n-1}(\frac{c_1}{2} R) p = c_3 R^{-1},$$

which can be made as small as we want with large enough $R$. □

**Remark.** A proof that applies for $-n < p < -n + 1$ was given by [CJQW, Prop. 4.2].

**References**


[CCF] Chlebík, Miroslav; Cianchi, Andrea; Fusco, Nicola. The perimeter inequality under Steiner symmetrization: Cases of equality, Ann. of Math. 162 (2005), 525-555.


[EMMP] Engelstein, Max; Marucchio, Anthony; Maurmann, Quinn; Pritchard, Taryn. Isoperimetric problems on the sphere and on surfaces with density, New York J. Math. 15 (2009), 97-123.


[MM] Maurmann, Quinn; Morgan, Frank. Isoperimetric comparison theorems for manifolds with density, Calc. Var. PDE 36 (2009), 1-5.


Alexander Díaz, Department of Mathematical Sciences,
University of Puerto Rico, Mayagüez, PR 00681
E-mail address: alexander.diaz1@upr.edu

Nate Harman, Department of Mathematics and Statistics,
University of Massachusetts, Amherst, MA 01003
E-mail address: nateharman1234@yahoo.com

Sean Howe, Department of Mathematics,
University of Arizona, Tucson, AZ 85721
E-mail address: seanpkh@gmail.com

David Thompson, Department of Mathematics and Statistics,
Williams College, Williamstown, MA 01267
E-mail address: dat1@williams.edu

Mailing Address: c/o Frank Morgan, Department of Mathematics and Statistics,
Williams College, Williamstown, MA 01267
E-mail address: Frank.Morgan@williams.edu