COMPACTNESS

Frank Morgan[1]

May, 2008

Abstract

In my opinion, compactness is the most important concept in mathematics. We'll track it from the one-dimensional real line in calculus to infinite dimensional spaces of functions and surfaces and see what it can do.

2000 Mathematics Subject Classification: 54D30

Keywords: *Compactness, Bolzano-Weierstrass, Alaoglu, soap films, geometric measure theory.*

1. *Department of Mathematics and Statistics, Williams College, USA.*

1 Compactness as the Universal Strategy

The universal problem in applied mathematics is to find an optimal solution. For example, the premier application of calculus is to solve max-min problems. Economists seek optimal economic strategies. Soap films as in Figure 1 seek the least-area way to span a wire boundary.

Figure 1: Soap films seek the least-area way to span a wire boundary, Photo by S. Devadoss.

The universal problem in pure mathematics is to prove that an optimal solution exists. Such a proof helps the applied problem-solver in two ways. First, it is a guarantee that a life-long search for a solution is not doomed to fail. Second, it is what makes many methods of solution work. For example, in calculus, the reason that you have to check only critical points (where the derivative is 0 fails to exist) and endpoints is that the theory guarantees that a continuous function *has* a maximum (and that a maximum must be a critical point or an endpoint).

The general strategy for proving that an optimal solution exists goes like this. Take a sequence of candidates with values approaching an optimum and get an ideal limit somehow. It is compactness that guarantees a limit.

This general strategy sometimes fails. For example, there is no largest positive number, because the positive numbers are not *bounded* (see Figure 2). Moreover, there is no smallest positive number, because the positive numbers are not *closed*, zero is not included. Sometimes compactness fails.

Figure 2: There is no largest positive number because the positive numbers are not bounded. There is no smallest positive number because the positive numbers are not closed

2 The Bolzano-Weierstrass Compactness Theorem of 1840

The first major result guaranteeing limits of sequences was proved by Karl Weierstrass in about 1840:

2.1 Bolzano-Weierstrass Theorem. *If* $S \subset \mathbf{R}^n$ *is bounded and closed, then every sequence in* S *has a convergent subsequence, converging to a limit in* S.

A set S with this property is called *compact* (or now sometimes "sequentially compact", to distinguish it from other definitions which can differ on complicated spaces). Here are three sample corollaries.

2.2 Corollary. *A continuous function* f *on* $[a, b] \subset \mathbf{R}$ *(or any closed and bounded subset of* $\mathbf{R}^n$*) has a minimum (and therefore calculus works).*

Proof sketch. Take a sequence of points x_i in $[a, b]$ with values $f(x_i)$ approaching the infimum. By Bolzano-Weierstrass, some subsequence converges to a point x in $[a, b]$. Because f is continuous, it hits its minimum at x.

2.3 Corollary. *There exists a least-perimeter n-gon of unit area in* $\mathbf{R}^2$.

Proof sketch. Since an n-gon is determined by n vertices, each with two coordinates, the set of n-gons is a subset of $\mathbf{R}^{2n}$. Moreover, the set

$$S = \{\text{n-gons through the origin with unit area with perimeter at most 100}\}$$

is a closed and bounded subset of $\mathbf{R}^{2n}$. Since perimeter is a continuous function on S, it has a minimum.

2.4 Corollary. The Isoperimetric Theorem. *The circle minimizes perimeter for given area in the plane.*

Proof sketch. By approximation, it suffices to prove that a perimeter-minimizing n-gon is regular. Zenodorus gave a beautiful, simple proof about 200 BC (see [1]), except that he had to assume that a perimeter-minimizing n-gon exists, so Weierstrass usually gets the credit.

Failure. Unfortunately, the Bolzano-Weierstrass Theorem fails in many spaces different from $\mathbf{R}^n$. For example, in the universe of rational numbers, the sequence which converges to $\sqrt{2}$ in $\mathbf{R}$,

$$1,\ 1.4,\ 1.41,\ 1.414, \ldots$$

has no convergent subsequence. As a second example, in what I'll call $\mathbf{R}^\infty$ (an infinite-dimensional real vector space), the sequence of unit basis vectors

$$e_1,\ e_2,\ e_3, \ldots$$

has no convergent subsequence. Likewise, the theorem fails for the space of possible soap films, which also is infinite dimensional.

3 Compactness in Metric Spaces

There is a generalization of Bolzano-Weierstrass from $\mathbf{R}^n$ to any finite or infinite-dimensional space with a distance or metric.

3.1 Compactness Theorem for Metric Spaces. *If S in a metric space is* totally bounded *and* complete, *then every sequence has a convergent subsequence, converging to a limit in S.*

Here *totally bounded* means that given $\varepsilon > 0$, S can be covered by finitely many balls of radius ε. *Complete* means that there are no holes, technically that every Cauchy sequence converges. In the example above, the unit ball about the origin is not totally bounded: take $\varepsilon = 1/2$ and note that each unit basis vector e_i is distance greater than 1 from each other e_j, so that covering them takes infinitely many such balls of radius ε.

Here are two big corollaries, starting with a compactness theorem for an infinite-dimensional space of functions.

3.2 Corollary. *The set of Lipschitz functions f from [0,1] into a ball in* $\mathbf{R}^n$ *is compact (under uniform convergence).*

Here f Lipschitz means that for some fixed constant C,

$$|f(y) - f(x)| \leq C\,|y - x|, \tag{1}$$

essentially that the functions have uniformly bounded derivative or velocity. That the main hypothesis to apply Theorem 3.1, total boundedness, follows from the Lipschitz condition, is a special case of Ascoli's Theorem (which applies to more general "equicontinuous" families of functions, such as "Hölder" functions).

3.3 Corollary. *On a smooth compact m-dimensional surface K in* $\mathbf{R}^n$*, there is a shortest path between any two points.*

Proof. We define a path as a function from $[0, 1]$ to K of constant velocity. Given two points, choose C such that some paths have velocity less than C. Corollary 3.2 implies that the set of such paths is compact, and it follows that there is a shortest one.

Failure for soap films. Unfortunately, for candidate soap films of small area with a given boundary, there is no immediate bound (1), because a skinny tentacle of negligible area as in Figure 3 can require a huge constant C.

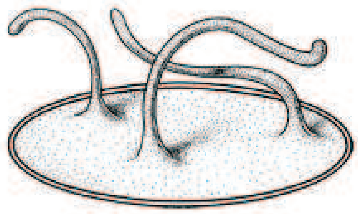

Figure 3: Long, skinny tentacles can carry an area-minimizing sequence of surfaces out of a compact space of Lipschitz functions. Figure by J. Bredt [[2], Fig. 1.3.3] © Frank Morgan.

4 The Hausdorff Metric

The following compactness theorem sounds like it would solve all our problems.

4.1. Theorem. *The space of all nonempty closed subsets of a ball in* $\mathbf{R}^n$ *is compact under the Hausdorff metric.*

The distance in the *Hausdorff metric* between any two closed sets is the farthest that any point of one set is from the other set. For example, the distance between the two faces of Figure 4 if placed in the same oval is the distance from the right eye of the first to the right eye of the second. Assuming that beauty is a continuous function, it follows from Theorem 4.1 that there is a most beautiful face.

Theorem 4.1 follows from a version of Corollary 3.2 by identifying a closed set A with the Lipschitz function "distance from A."

Gromov generalized the Hausdorff metric from two subsets of $\mathbf{R}^n$ to two abstract metric spaces A, B, by taking an infimum over all isometric embeddings of A and B into other metric spaces. Cheeger used the Gromov-Hausdorff metric to prove that bounds on curvature, diameter, and volume limit n-dimensional manifolds to finitely many diffeomorphism types.

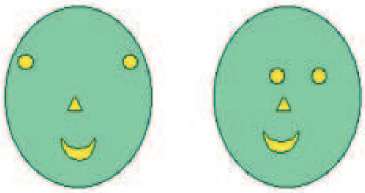

Figure 4: The Hausdorff distance between two sets is the farthest that any point of one set is from the other set. The distance between these two sets of facial features if placed in the same oval is the distance from the right eye of the first to the right eye of the second.

Failure for soap films. Unfortunately, Theorem 4.1 does not produce area-minimizing soap films, because in this weak topology area is not continuous. An area-minimizing sequence of surfaces growing lots of long, skinny tentacles of vanishing area can yield a solid volume of infinite area in the limit, as in Figure 5.

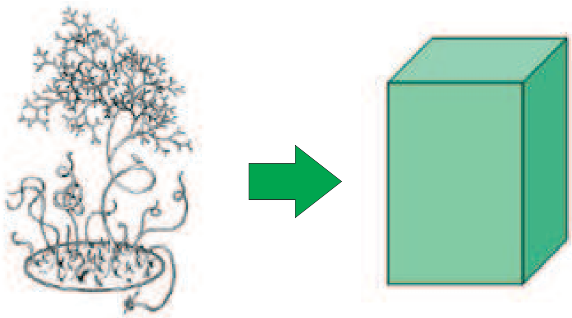

Figure 5: Long, skinny tentacles on this area-minimizing sequence of surfaces produce a solid block of infinite area in the Hausdorff metric limit. Left figure by J. Bredt [[2], Fig. 1.3.4] © Frank Morgan.

5 Alaoglu's Theorem of 1940

In addition to the Hausdorff metric, there is another weak topology defined on the dualspace of a normed vectorspace, the weak* ("weak-star") topology, which provides a compactness theorem more respectful of the size of sets. The *dualspace* is defined as the space of all bounded linear functions. For example, $\mathbf{R}^n$ with basis $e_1, e_2, \ldots, e_n$ has dualspace $\mathbf{R}^{n*}$ with basis $e_1^*, e_2^*, \ldots, e_n^*$, where e_i^* is the linear function on $\mathbf{R}^n$ which just picks out the i^{th} coordinate. A sequence of such functions f_n converges to a limit f in the weak* topology if for every x in the original vectorspace, the sequence of numbers $f_n(x)$ converges to $f(x)$.

5.1. Alaoglu's Theorem. *The unit ball in the dualspace of a normed vectorspace is weak* compact.*

At first this may seem to contradict our earlier counterexample to compactness in infinite-dimensional vectorspaces. Consider for example the normed vectorspace l^2 of real sequences (x_i) of finite norm:

$$|(x_i)|^2 = \Sigma x_i^2 < \infty.$$

The unit dual basis elements

$$e_1^*,\ e_2^*,\ e_3^*, \ldots$$

seem to provide a counterexample, a sequence with no convergent subsequence. The resolution is that in this new weak* topology, this sequence *does* converge, to 0. Indeed, for any vector (x_i),

$$e_n^*(x_i) = x_n \to 0.$$

In analysis, this kind of convergence is called "pointwise convergence". In topology, it is called convergence in the "product topology." Indeed, Alaoglu's Theorem follows from Tychonoff's Theorem in topology, which says that if K is compact, then the set K^X of all functions from X to K is compact in the product topology.

5.2. Corollary. *The set of bounded measures on a ball B in* $\mathbf{R}^n$ *is compact.*

Proof. Bounded measures constitute the dualspace to the space of continuous functions on B. (Weak* convergence is determined by the convergence of the integral of every continuous function with respect to the measures.)

Near-miss for soap films. Corollary 5.2 is very promising for applications to soap films, because a two-dimensional surface S in $\mathbf{R}^3$ can be viewed as a measure μ_S on $\mathbf{R}^3$, with the help of the standard two-dimensional "Hausdorff" measure $\mathcal{H}^2$ on $\mathbf{R}^3$, as follows. Given a surface S of finite area, for any set A such as a small ball in $\mathbf{R}^n$, define

$$\mu_S(A) = \mathcal{H}^2\ (A \cap S).$$

Because it ignores sets of vanishing measure, this approach avoids the pitfalls of Figure 5. The only trouble is that the space of measures is too big: it includes smeared out surfaces and probability distributions. We want to be guaranteed a real surface as our area-minimizing soap film.

6 The Compactness Theorem of Geometric Measure Theory

In the 1960s, Federer and Fleming introduced the ultimate compactness theorem. It applies to a very satisfactory class of "rectifiable" surfaces, general enough to include anything anyone would call a surface, allowing singularities and infinite topological complexity, but nothing more. Roughly, m-dimensional rectifiable surfaces are defined as countable unions of Lipschitz images of subsets of $\mathbf{R}^m$. Convergence means weak* convergence as measures, or equivalently, convergence in a more geometric "flat norm."

6.1. The Compactness Theorem of Geometric Measure Theory. *The set of all m-dimensional rectifiable surfaces in a ball in* $\mathbf{R}^n$*, with a fixed bound on area and perimeter, is compact.*

The proof shows that this space of rectifiable surfaces is closed in the space of measures. This theorem finally solves the soap film problem:

6.2. Corollary. Solution of the Soap Film Problem. *Given a closed curve C of finite length in* $\mathbf{R}^3$*, there is a rectifiable surface of least area bounded by the given curve.*

Proof. The curve C lies in some large ball B. Since projection to the ball does not increase area, we may confine attention to surfaces in the ball, of fixed perimeter and bounded area. There is some (perhaps singular) surface with that boundary, for example, a cone over the boundary. Take a sequence of surfaces with areas approaching the infimum. By the Compactness Theorem 6.1, some subsequence converges to a limit surface S. Since in general the area of a limit cannot exceed the lim inf of the areas, S is the desired surface of least area.

6.3. Remarks. In 1930 Jesse Douglas received the first Fields Medal for a real analysis proof of the existence of least-area surfaces, but the definition used by him and his successors, with restrictions on topological type, admitted self-intersection singularities that cannot occur in soap films. On the other hand, the area-minimizing rectifiable surfaces of geometric measure theory, with given smooth boundary, turn out to be smooth, embedded manifolds.

6.4. Open Question. In fact, there are still more general physical soap films which have the given boundary in a weaker sense, as in Figure 6. For the broadest such class, it remains an open question today whether there is one of least area, or whether the limit might be an unstable non-soap-film that would collapse onto the boundary.

Acknowledgements. Quiero agradecer a Cesar Silva y sus colegas por la invitación a visitar la Universidad Católica del Perú y enviar este artículo a Pro Mathematica.

This article is based on a talk at the Williams College undergraduate mathematics colloquium. For more information, see Morgan's book on Geometric Measure Theory, 2008 edition. This work was partially supported by a grant from the National Science Foundation.

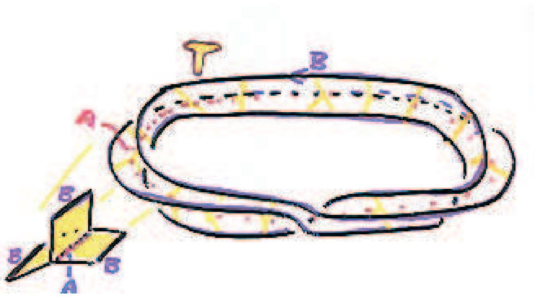

Figure 6: If soap films are allowed extra internal boundary, like the dashed curve A where this soap film meets in threes, it is an open question whether there is a soap film of least area.

References

[1] Hugh Howards, Michael Hutchings, and Frank Morgan, The isoperimetric problem on surfaces, Amer. Math. Monthly 106 (1999), 430-439.

[2] Frank Morgan, Geometric Measure Theory, Academic Press, 4th ed., 2008.

Resumen

En mi opinión, la compacidad es el concepto más importante de las matemáticas. La seguiremos desde la línea unidimensional en cálculo hasta los espacios de dimensión infinita de funciones y de superficies y veremos lo que puede hacer.

Palabras Clave: Compacidad, Bolzano-Weierstrass, Alaoglu, películas de jabón, teoría geométrica de la medida

Frank Morgan
Department of Mathematics and Statistics Williams College
Williamstown, Massachusetts 01267
frank.morgan@williams.edu