RIGIDITY FOR NONNEGATIVELY CURVED METRICS ON $S^2 \times \mathbb{R}^3$

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Abstract. We address the question: how large is the family of complete metrics with nonnegative sectional curvature on $S^2 \times \mathbb{R}^3$? We classify the connection metrics, and give several examples of non-connection metrics. We provide evidence that the family is small by proving some rigidity results for metrics more general than connection metrics.

1. Introduction

According to [2], the space $S^2 \times \mathbb{R}^2$ with an arbitrary complete metric of nonnegative sectional curvature must be a product metric or be isometric to a Riemannian quotient of the form

$$((S^2, g_0) \times (\mathbb{R}^2, g_f) \times \mathbb{R})/\mathbb{R},$$

were g_0 and g_f are $\mathbb{R}$-invariant metrics. These metrics are "rigid at the soul", meaning the following inequality from [7] becomes an equality in the case when M is diffeomorphic to $S^2 \times \mathbb{R}^2$:

Proposition 1.1. [7] *If M is an open manifold of nonnegative curvature with soul Σ, then for any $p \in \Sigma$, orthonormal vectors $X, Y \in T_p\Sigma$, and orthonormal $W, V \in \nu_p(\Sigma)$ = the normal space to $T_p\Sigma$ in T_pM,*

$$\langle (D_X R)(X,Y)W,V\rangle^2 \leq (|R(W,V)X|^2 + (2/3)(D_{X,X}R)(W,V,V,W)) \cdot R(X,Y,Y,X).$$

In general, the metric sphere, S_ϵ, of small radius ϵ about a soul inherits a metric of nonnegative curvature [4]. If the above inequality is strict, then S_ϵ is positively curved [7]. If there is a single $(p,W) \in \nu(\Sigma)$ such that the inequality is strict for all X, Y, V at p, then ϵW is a point of S_ϵ with positive curvature, so S_ϵ is quasi-positively curved, which means that it has nonnegative curvature, and positive curvature at a point (in this case, we will say that the inequality is "quasi-strict").

The question whether $S^2 \times S^2$ admits positive or quasi-positive curvature is a long-standing unsolved problem in Riemannian geometry, which motivates our study of metrics of nonnegative curvature on $S^2 \times \mathbb{R}^3$. Although we do not completely classify such metrics, we demonstrate special cases under which the inequality rigidly determines what the metrics may look like at the soul, which provides evidence that the family of metrics is small.

We first classify the connection metrics, i.e., metrics with totally geodesic Sharafutdinov fibers. It is convenient to simultaneously study the nontrivial $\mathbb{R}^3$ bundle over S^2, whose associated sphere bundle has total space $CP^2 \# \overline{CP^2}$.

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Proposition 1.2. *Every nonnegatively curved connection metric on an $\mathbb{R}^3$-bundle over S^2 is isometric to a Riemannian quotient of the form:*

$$M = ((S^3, g_0) \times (\mathbb{R}^3, g_f))/S^1,$$

where g_f is an S^1-invariant metric on $\mathbb{R}^3$ and g_0 is a connection metric on the principal bundle $S^1 \hookrightarrow S^3 \to S^2$.

Notice that an integer k determines the relative speeds at which S^1 acts on S^3 and $\mathbb{R}^3$, and the bundle is trivial if and only if k is even. The proposition implies that the holonomy group of the normal bundle of the soul (the "normal holonomy group") is either trivial or isomorphic to S^1. The proposition is actually a corollary of a similar fact for S^2-bundles:

Proposition 1.3. *For an S^2-bundle $\pi : M \to B = S^2$, suppose that M and B have nonnegatively curved metrics so that π is a Riemannian submersion with totally geodesic fibers. Then $\pi : M \to B$ can be metrically re-described as*

$$M = ((S^3, g_0) \times (S^2, g_1))/S^1 \xrightarrow{\pi} (S^3, g_0)/S^1,$$

where g_0 is a connection metric on the principal bundle $S^1 \hookrightarrow S^3 \to S^2$, and g_1 is an S^1-invariant metric on S^2.

In the both propositions, g_0 is a connection metric on S^3, which means that (1) the principal S^1 action is by isometries, and (2) all it's orbits have the same length. If instead we assume only (1), then g_0 is called a "warped connection metric", and the resulting metric on the $\mathbb{R}^3$ (or S^2) bundle over S^2 is a non-connection metric with holonomy group S^1. However, not all nonnegatively curved metrics with holonomy group S^1 are of this type, as the next example shows.

Example 1.4. In Proposition 1.2, if g_f is a product metric, i.e., $(\mathbb{R}^3, g_f) = (\mathbb{R}^2, g_1) \times \mathbb{R}$, then

$$M = ((S^3, g_0) \times (\mathbb{R}^3, g_f))/S^1 = (((S^3, g_0) \times (\mathbb{R}^2, g_1))/S^1) \times \mathbb{R} = (E, g_E) \times \mathbb{R},$$

where (E, g_E) is the total space of an $\mathbb{R}^2$-bundle over S^2 with a connection metric. Walschap showed in [8, Theorem 2.1] that when E is nontrivial ($k \neq 0$), such a connection metric g_E can be non-rigid. The most natural one, coming from the round g_0 and the flat g_1, can be altered fairly arbitrarily away from the soul without losing nonnegative curvature. If $g_{E'}$ is such an alteration, then $(\mathbb{R}^2, g_{E'}) \times \mathbb{R}$ has nonnegative curvature and normal holonomy group S^1. Considering small metric spheres about a soul of $(\mathbb{R}^2, g_{E'}) \times \mathbb{R}$ produces a large family of nonnegatively curved metrics on $S^2 \times S^2$ (k even) and on $CP^2 \# \overline{CP^2}$ (k odd).

Although the metrics is the previous example are flexible, they are also rigid in the sense that the inequality of Proposition 1.1 is not quasi-strict, which follows from the following:

Proposition 1.5. *For an S^2-bundle $\pi : M \to B = S^2$, suppose that M and B have nonnegatively curved metrics so that π is a Riemannian submersion with holonomy group S^1. Then through every point of M there passes a one parameter family of totally geodesic flat strips.*

The remaining case is when the normal holonomy group is transitive, and we begin by showing that examples of this phenomenon exist:

Proposition 1.6. *Consider the following nonnegatively curved metric on $S^2 \times \mathbb{R}^3$:*

$$M = ((S^2, round) \times (\mathbb{R}^3, g_f) \times (SO(3), g_B))/SO(3),$$

where g_f is a nonnegatively curved $SO(3)$-invariant metric on $\mathbb{R}^3$ and g_B is a bi-invariant metric on $SO(3)$. Then the normal holonomy group of M is $SO(3)$.

Notice that the right $SO(3)$-action on the third factor of $S^2 \times R^3 \times SO(3)$ induces an isometric action of $SO(3)$ on M. It follows that the soul of M is round and that the Sharafutinov fibers of M have mutually isometric S^1-invariant metrics.

If g_B is replace by a right-invariant nonnegatively curved metric in this construction, then M is still nonnegatively curved. These are the only known examples with transitive holonomy. In particular, for the non-trivial $\mathbb{R}^3$-bundle over S^2, it is not known whether there is a nonnegatively curved metric with transitive normal holonomy group. Although it is difficult to describe the general right-invariant case as explicitly as we describe the bi-invariant case, it is at least straightforward to check that the metric spheres about the soul are not quasi-positively curved, so the inequality of Proposition 1.1 is not quasi-strict.

Since there are no other known examples with with transitive holonomy, one might ask whether an arbitrary metric with transitive holonomy looks like with one of these examples, at least at the soul. In section 6, we show some special cases in which the inequality of Proposition 1.1 rigidly determines what the metric may look like at the soul. For example, if we assume that the Sharafutdinov fibers are all S^1-invariant, then the inequality is not quasi-strict. If we additionally assume that the connection in the normal bundle of the soul has a special form, then the the soul must be round. These added assumptions are motivated by the geometry of the known examples, and our rigidity statements provide evidence that the family of nonnegatively curved metrics on $S^2 \times \mathbb{R}^3$ with transitive holonomy is small.

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2. Connection metrics

In this section, we prove Proposition 1.2, which classifies the nonnegatively curved connection metrics on an $\mathbb{R}^3$-bundle over S^2.

Lemma 2.1. *For a connection metric of nonnegative curvature on an $\mathbb{R}^3$-bundle over S^2, the normal holonomy group is trivial or is isomorphic to S^1.*

Proof. Let M be the space $S^2 \times \mathbb{R}^3$ together with a nonnegatively curved connection metric. Let Σ be a soul of M. Let $p \in \Sigma$, and let X, Y be an orthonormal basis of $T_p\Sigma$. Let R^∇ denote the curvature tensor of the connection in the normal bundle of Σ.

The map $U \to R^\nabla(X,Y)U$ is a skew-symmetric endomorphism of the fiber $\nu_p(\Sigma) = \mathbb{R}^3$, so there must be a non-zero vector W in it's kernel. Choose any $V \in \nu_p(\Sigma)$ such that $\{V, W\}$ is orthonormal. Let $\alpha : [0,a] \to \Sigma$ be a piecewise-geodesic loop at p. Let X_t, Y_t, W_t, V_t denote the parallel transports of X, Y, W, V along α. From Proposition 1.1,

$$\langle (D_{\alpha'(t)} R^\nabla)(X_t, Y_t)W_t, V_t\rangle^2 \leq |R^\nabla(W_t, V_t)X_t|^2 \cdot R(X_t, Y_t, Y_t, X_t).$$

In other words, if we let $f(t) = |R^\nabla(W_t, V_t)X_t| = \langle R^\nabla(X_t, Y_t)W_t, V_t\rangle$, then

$$f'(t)^2 \leq f(t)^2 \cdot R(X_t, Y_t, Y_t, X_t).$$

Since W is chosen so that $f(0) = 0$, it follows that $f(t) = 0$ for all t. So either W is fixed by the action of the normal holonomy group on $\nu_p(\Sigma)$, or $R^\nabla = 0$, which implies that the normal holonomy group is trivial. $\square$

The following would be a natural generalization of Lemma 2.1

Conjecture 2.2. For a connection metric of nonnegative curvature on an $\mathbb{R}^k$-bundle over a manifold M^n, if $k > n$ then the normal holonomy group acts reducibly.

Next we prove a result for S^2 bundles over S^2 analogous to Lemma 2.1.

Lemma 2.3. *For an S^2-bundle $\pi : M \to B = S^2$, suppose that M and B have non-negatively curved metrics so that π is a Riemannian submersion with totally geodesic fibers. Then the holonomy group of π is trivial or is isomorphic to S^1.*

Proof. Let $\mathcal{H}, \mathcal{V}$ denote the horizontal and vertical distributions of π, and let A denote the A-tensor of π. Since $\mathcal{H}$ is 2-dimensional, $\mathcal{U}_p = A(X, Y)$ does not depend on the choice of oriented orthonormal basis $\{X, Y\}$ of $\mathcal{H}_p$. Thus $\mathcal{U}$ is a well-defined global vertical vector field on M.

Let $p \in M$ be a point where $\mathcal{U}_p = 0$. There must be such a point on each π-fiber, since S^2 does not admit a nowhere vanishing vector field. Let X, Y be an oriented orthonormal basis for $\mathcal{H}_p$, and let $V \in \mathcal{V}_p$ be arbitrary. Using O'Neill's formula for Riemannian submersions,

$$\begin{aligned} R(X, Y+V, Y+V, X) &= R(X, Y, Y, X) + 2R(X, Y, V, X) + R(X, V, V, X) \\ &= R(X, Y, Y, X) + 2\langle (D_X A)_X Y, V\rangle + 0 \\ &= R(X, Y, Y, X) + 2\langle \nabla_X \mathcal{U}, V\rangle \geq 0 \end{aligned}$$

Since the above is true for any V (of any length), it follows that $(\nabla_X \mathcal{U})^{\mathcal{V}} = 0$. It's also easy to see that $(\nabla_X \mathcal{U})^{\mathcal{H}} = 0$ since for any $Z \in \mathcal{H}_p$, $\langle \nabla_X \mathcal{U}, Z\rangle = \langle A_X U, Z\rangle = 0$.

Since the fibers are totally geodesic, $\mathcal{U}$ restricted to any fiber is a Killing field. So, assuming $\mathcal{U}$ does not vanish on the entire fiber F_p through p, p is an isolated zero of $\mathcal{U}$ on F_p. In a neighborhood of p in M, the set S of zeros of $\mathcal{U}$ is a 2-dimensional smooth submanifold. But,

$$T_p S \subset \{X \in T_p M | \nabla_X \mathcal{U} = 0\} = \mathcal{H}_p.$$

So $T_p S = \mathcal{H}_p$, and S is an integral submanifold of $\mathcal{H}$ near p. This implies that p is a fixed point of the holonomy group. $\square$

Proof of Proposition 1.3. Choose a fiber of $\pi : M \to B = S^2$. Define $\mathcal{U}$ as in the previous proof. $\mathcal{U}$ restricted to this fiber is a Killing field. Let O denote an orbit of maximal length (the "equator" of the fiber). Let N denote the orbit of O under the holonomy group, which is a smooth 3-dimensional submanifold of M (the union of the equators of all of the fibers). It is straightforward to verify that N is totally geodesic.

The induced metric on the circle bundle $S^1 \hookrightarrow N \overset{\pi|_N}{\to} B$ is a connection metric, which we claim can be metrically re-described as:

$$N = ((S^3, g_0) \times S^1(r))/S^1 \overset{\pi|_N}{\to} B = (S^3, g_0)/S^1.$$

More precisely, the relative speed, k, at which S^1 acts on S^3 and S^1 can be first chosen to give the topologically correct bundle. Next, choosing a connection metric g_0 on the bundle $S^1 \hookrightarrow S^3 \to S^2$ means choosing (1) a metric on S^2, (2) a principal connection in the bundle, (3) the fiber-length, l. We make the first choice such that $(S^3, g_0)/S^1$ is isometric to B. We make the second choice to induce the correct horizontal distribution in the circle bundle. Finally, we can choose l and r together so that the fiber-length is correct.

If g_1 denotes any S^1-invariant metric on S^2 whose equator (meaning the maximal-length orbit of the S^1-action) has circumference $2\pi r$, then the Riemannian submersion

$$\tilde{M} = ((S^3, g_0) \times (S^2, g_1))/S^1 \xrightarrow{\tilde{\pi}} B = (S^3, g_0)/S^1$$

has totally geodesic fibers and holonomy group S^1. Further, if $\tilde{N}$ denotes the union of the equators of all of the fibers of $\tilde{\pi}$, then $\tilde{N} \xrightarrow{\tilde{\pi}|_{\tilde{N}}} B$ is metrically equal to $N \xrightarrow{\pi|_N} B$. Finally, g_1 can easily be chosen so that the fiber metric of $\tilde{\pi}$ agrees with the fiber metric on π. It's then straightforward to see that $\tilde{M} \xrightarrow{\tilde{\pi}} B$ is metrically equal to $M \xrightarrow{\pi} B$. $\square$

Proof of Proposition 1.2. For a nonnegatively curved connection metric on an $\mathbb{R}^3$-bundle, M, over S^2, the distance sphere, S_r, of any radius r about a soul inherits a metric which, by Proposition 1.3, can be described as:

$$S_r = ((S^3, g_0) \times (S^2, g_1))/S^1.$$

As we vary r, the soul metric $(S^3, g_0)/S^1$ does not change, and neither does the horizontal distribution of $(S^3, g_0) \to (S^3, g_0)/S^1$, since horizontal displacement of points in the sphere bundles is controlled by parallel transport of vectors in the normal bundle of the soul. So we can choose g_0 and k independent of r. We write $g_1(r)$ to show the dependence of g_1 on r. Now define an S^1-invariant metric g_f on $\mathbb{R}^3$ such that the distance sphere of radius r about the origin has metric $g_1(r)$. It follows that M is isometric to $((S^3, g_0) \times (\mathbb{R}^3, g_f))/S^1$. $\square$

3. Non-connection metrics with normal holonomy group S^1

We saw in example 1.4 that there is no simple group-theoretic classification of nonnegatively curved metrics on $\mathbb{R}^3$-bundles over S^2 with normal holonomy group S^1. In this section, we prove Proposition 1.5, which says that even though such metrics are in one sense perturbable, there is also rigidity.

Proof of Proposition 1.5. Since the holonomy group is S^1, there exists a totally geodesic horizontal section $\iota : B \to M$. Since ι is an isometric embedding, one can consider $\iota \circ \pi$ as a distance non-increasing retraction from M to $\iota(B)$. Perelman's argument in [5] shows that every vertizontal plane based at $\iota(B)$ generates a totally geodesic immersed flat strip. $\square$

4. An example with transitive normal holonomy group

In this sections, we prove Proposition 1.6, which describes the geometry of the Riemannian manifold

$$M = ((S^2, \text{round}) \times (\mathbb{R}^3, g_f) \times (\text{SO}(3), g_B))/SO(3),$$

To simplify the discussion, we take the unit-round metric on S^2, we take g_f to be the flat metric on $\mathbb{R}^3$, and we assume the bi-invariant metric g_B is scaled so that a unit-length vector in so(3) corresponds to a Killing field on $S^2(1)$ with maximal norm one. Let Σ denote the soul of M. We begin by explicitly describing parallel transport in $\nu(\Sigma)$, proving in particular:

Lemma 4.1. *The normal holonomy group of M is $SO(3)$.*

Proof. Let g denote the product of the unit-round metric on S^2 with the flat metric on $\mathbb{R}^3$. Let $\tilde{g}$ denote the quotient metric on $S^2 \times \mathbb{R}^3$ obtained from the above description of M. According to [1], $\tilde{g}$ is obtained from g by "rescaling along the $SO(3)$ action." We next describe more precisely how the metric $\tilde{g}$ on $T_{(p,V)}(S^2 \times \mathbb{R}^3)$ is obtained from the metric g.

Let $(p,V) \in S^2(1) \times \mathbb{R}^3$ with $|V| = 1$. Using the natural inclusion $S^2(1) \subset \mathbb{R}^3$, define θ as the angle between p and V. Assume that $V \neq \pm p$. Denote:

$$W = \frac{V - \langle V,p\rangle \cdot p}{|V - \langle V,p\rangle \cdot p|} \quad \text{and} \quad U = W \times p,$$

where "$\times$" denotes the vector cross product in $\mathbb{R}^3$. Notice that $\{W,p,U\}$ is an oriented orthonormal basis of $\mathbb{R}^3$, and $V = \cos\theta \cdot p + \sin\theta \cdot W$. Next, consider the following orthonormal basis $\{X,A,Y,B,\hat{r}\}$ of $T_{(p,V)}(S^2 \times \mathbb{R}^3)$:

$$X = (-W,0), A = (U,0), Y = (0, \sin\theta \cdot p - \cos\theta \cdot W), B = (0,U), \hat{r} = (0,V).$$

Choose an orthonormal basis $\{E_1,E_2,E_3\}$ of so(3) corresponding to unit-speed right-handed rotations of S^2 about the vectors $\{U,p,W\}$ respectively. The values, $T_1,T_2,T_3 \in T_{(p,V)}(S^2 \times \mathbb{R}^3)$, of the corresponding Killing fields at (p,V) are:

$$T_1 = X + Y, \quad T_2 = \sin\theta \cdot B, \quad T_3 = A + \cos\theta \cdot B.$$

Define

$$K = (k_{ij}) = (\langle T_i, T_j\rangle_g) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sin^2\theta & \sin\theta\cdot\cos\theta \\ 0 & \sin\theta\cdot\cos\theta & 1+\cos^2\theta \end{pmatrix}.$$

The metric $\tilde{g}$ agrees with g on the orthogonal compliment of span$\{T_i\}$. On span$\{T_i\}$ we have:

$$\tilde{K} = \left(\tilde{k}_{ij}\right) = (\langle T_i, T_j\rangle_{\tilde{g}}) = K \cdot (I+K)^{-1} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 2\frac{\cos^2\theta - 1}{\cos^2\theta - 4} & -\frac{\sin\theta\cdot\cos\theta}{\cos^2\theta-4} \\ 0 & -\frac{\sin\theta\cdot\cos\theta}{\cos^2\theta-4} & \frac{-2}{\cos^2\theta-4} \end{pmatrix}.$$

It is useful to define $T_4 = X - Y$, so that $\{T_1,T_2,T_3,T_4,\hat{r}\}$ becomes a basis for $T_{(p,V)}(S^2 \times \mathbb{R}^3)$. The change of basis matrix which translates from $\{X,A,Y,B,\hat{r}\}$ to $\{T_1,T_2,T_3,T_4,\hat{r}\}$ is $N = \begin{pmatrix} \frac12 & 0 & \frac12 & 0 & 0 \\ 0 & -\frac{\cos\theta}{\sin\theta} & 0 & \frac{1}{\sin\theta} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac12 & 0 & -\frac12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. So if $\tilde{K}' = \begin{pmatrix} \tilde{K} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then two vector $\mathcal{X},\mathcal{Y} \in T_{(p,V)}(S^2 \times \mathbb{R}^3)$ written in the basis $\{X,A,Y,B,\hat{r}\}$ have inner product:

$$\langle \mathcal{X},\mathcal{Y}\rangle_{\tilde{g}} = (N\cdot\mathcal{X})^T \cdot \tilde{K}' \cdot (N \cdot \mathcal{Y}). \tag{4.1}$$

Since the metric in the $\hat{r}$ direction is unchanged when passing from g to $\tilde{g}$, the Sharafutdinov map $\pi : M \to \Sigma$ is also unchanged when passing from g to $\tilde{g}$. So the vertical space of π is unchanged:

$$\mathcal{V}_{(p,V)} = \text{span}\{Y,B,\hat{r}\} = T_V\mathbb{R}^3 \subset T_{(p,V)}(S^2 \times \mathbb{R}^3).$$

It is now easy to show that the horizontal space of π is

$$\mathcal{H}_{(p,V)} = \text{span}\{X + \frac12 \cdot Y, A + \frac12 \cos\theta \cdot B\}. \tag{4.2}$$

To verify this, use equation 4.1 to show that both of these vectors are $\tilde{g}-$perpendicular to Y, B, and $\hat{r}$.

Equation 4.2 allows a simple description of $\tilde{g}$-parallel transportation in $\nu(\Sigma)$. Namely, suppose that γ is a geodesic segment in Σ with $\gamma(0) = p$. Using the g-parallel identification $\mathbb{R}^3 = \nu_p(\Sigma) = \nu_{\gamma(t)}(\Sigma)$, we can think of the parallel transport of $V \in \nu_p(\Sigma)$ along γ as a path, $V(t)$, in $\mathbb{R}^3$. If $\mathcal{U} \in \Sigma$ is perpendicular to the plane in which γ lies, then $V(t)$ is the result of rotating V an angle of $t/2$ about the axis determined by $\mathcal{U}$. In other words, parallel transport rotates $\mathbb{R}^3$ along with γ, but at half the speed. The Lemma follows easily. $\square$

Lemma 4.2. *The soul is a round sphere with constant curvature* 2*.*

Proof. As before, let $T_1, T_2, T_3 \in T_{(p,0)}(S^2 \times \mathbb{R}^3)$ denote the values at $(p, 0)$ of the Killing fields associated with an orthonormal basis E_1, E_2, E_3 of so(3). If the E_i's are chosen as before, then $T_2 = 0$ and $\{T_1, T_3\}$ form a g-orthonormal basis of $T_pS^2 \subset T_{(p,0)}(S^2 \times \mathbb{R}^3)$. So $\langle T_1, T_3\rangle_{\tilde{g}} = 0$ and for $i = 1, 3$, $|T_i|_{\tilde{g}} = |T_i|_g/\sqrt{1 + |T_i|_g} = 1/\sqrt{2}$. In other words, the metric on the soul is the unit-round metric with the norms of all vectors rescaled by a factor of $1/\sqrt{2}$. This metric has constant curvature 2. $\square$

As mentioned in the introduction, the right $SO(3)$ action on the third factor of $S^2 \times \mathbb{R}^3 \times SO(3)$ induces an isometric $SO(3)$-action on M. It follows that the intrinsic metric on the Sharafutdinov fiber over $p \in \Sigma$ is S^1-invariant. The fixed-vector of this S^1-symmetry is p (as before, we are considering $p \in \mathbb{R}^3$ in the natural way). Any two such fibers are isometric. The S^1-symmetry implies that the curvature of a vertical 2-plane σ at p depends only the angle θ that σ makes with p. We leave it to the reader to verify that planes containing p have curvature 3, while the plane orthogonal to p has curvature $3/2$, so

$$k(\sigma) = 3\sin^2\theta + \frac{3}{2}\cos^2\theta. \tag{4.3}$$

Next we describe the curvature tensor R^∇ of the connection ∇ in the normal bundle of the soul.

Lemma 4.3. *Let $p \in \Sigma$, let $\{X, Y\}$ be an oriented orthonormal basis of $T_p\Sigma$, and let $V \in \nu_p(\Sigma)$. Letting "$\times$" denote the vector cross product in $\mathbb{R}^3$, and using the natural identification $\Sigma = S^2 \subset \mathbb{R}^3 = \nu_p(\Sigma)$, we have:*

$$R^\nabla(X, Y)V = \frac{3}{2}V \times p.$$

In particular, $|R^\nabla| = \frac{3}{2}$ and $R^\nabla(X, Y)p = 0$.

Although we could prove Lemma 4.3 directly from equation 4.2, we find it simpler and more illuminating to prove a more general formula in the next section.

5. A family of connections in a trivial $\mathbb{R}^3$-bundle

In this section, we study a natural family of connections in the trivial $\mathbb{R}^3$-bundle over a manifold B^n.

Let $f : B^n \to S^2$ denote any smooth function. Let λ be any real number. There is a connection, ∇, in the trivial vector bundle $B^n \times \mathbb{R}^3$ naturally associated with $\{f, \lambda\}$ as follows. Let $\Phi : TS^2 \to so(3)$ denote the canonical map, which can be described algebraically as

$$\Phi(p, X)(W) = (p \times X) \times W = \langle p, W\rangle X - \langle X, W\rangle p.$$

Then $\mathcal{D} = \Phi \circ df : TB^n \to so(3)$ is a connection difference form; that is, $\mathcal{D}$ is a $so(3)$-valued one-form on B^n. So if $\overline{\nabla}$ denotes the flat connection on $B^n \times \mathbb{R}^3$, then

$$\nabla = \overline{\nabla} + \lambda\mathcal{D}$$

is a connection on $B^n \times \mathbb{R}^3$.

Lemma 5.1. *The connection in the normal bundle of the soul of*

$$((S^2, round) \times (\mathbb{R}^3, g_f) \times (SO(3), g_B))/SO(3)$$

is determined as above by the identity map $f : S^2 \to S^2$, with $\lambda = -1/2$.

Proof. This follows immediately from equation 4.2. □

Thinking of f as a unit-length section of the bundle, the next lemma says that the curvature tensor, R^∇, of ∇ vanishes along f, and the norm of R^∇ depends on the area-distortion of f. Notice that Lemma 4.3 is a corollary of Lemma 5.2 and Lemma 4.2.

Lemma 5.2. *Let $p \in B^n$, $X, Y \in T_pB^n$ and $W \in \mathbb{R}^3$. Define $\overline{X} = df_pX$ and $\overline{Y} = df_pY$. Then*

$$R^\nabla(X,Y)W = \lambda(\lambda+2)\left(\langle\overline{X},W\rangle\overline{Y} - \langle\overline{Y},W\rangle\overline{X}\right).$$

In particular, $R^\nabla(X,Y)f(p) = 0$ and $|R^\nabla(X,Y)| = |\lambda^2 + 2\lambda| \cdot |\overline{X} \wedge \overline{Y}|$.

Proof. Extend $\{X, Y\}$ to local vector fields on B^n which commute at p. Extend W to a $\overline{\nabla}$-parallel section of the bundle. Let $\overline{p} = f(p)$. Then

$$\begin{aligned} R^\nabla(X,Y)W &= \nabla_X\nabla_Y W - \nabla_Y\nabla_X W \\ &= \lambda(\nabla_X\mathcal{D}_Y W - \nabla_Y\mathcal{D}_X W) \\ &= \lambda(\overline{\nabla}_X\mathcal{D}_Y W - \overline{\nabla}_Y\mathcal{D}_X W) + \lambda^2(\mathcal{D}_X\mathcal{D}_Y W - \mathcal{D}_Y\mathcal{D}_X W). \end{aligned}$$

We have at p that:

$$\mathcal{D}_Y W = (\overline{p} \times \overline{Y}) \times W = \langle\overline{p},W\rangle\overline{Y} - \langle\overline{Y},W\rangle\overline{p},$$

so

$$\mathcal{D}_X\mathcal{D}_Y W = \langle\overline{p},\mathcal{D}_Y W\rangle\overline{X} - \langle\overline{X},\mathcal{D}_Y W\rangle\overline{p} = -\langle\overline{Y},W\rangle\overline{X} - \langle\overline{p},W\rangle\langle\overline{X},\overline{Y}\rangle p,$$

and therefore,

$$\mathcal{D}_X\mathcal{D}_Y W - \mathcal{D}_Y\mathcal{D}_X W = \langle\overline{X},W\rangle\overline{Y} - \langle\overline{Y},W\rangle\overline{X}.$$

Next let $p(t)$ denote a path in B^n with $p(0) = p$ and $p'(0) = X$. It is convenient to let $\bar{p}(t) = f(p(t))$ and $\overline{Y}(t) = df_{p(t)}(Y(p(t)))$. then,

$$(\mathcal{D}_Y W)_{p(t)} = (\bar{p}(t) \times \overline{Y}(t)) \times W = \langle\bar{p}(t),W\rangle\overline{Y}(t) - \langle\overline{Y}(t),W\rangle\bar{p}(t)$$

So at p,

$$\begin{aligned} \overline{\nabla}_X\mathcal{D}_Y W &= \left.\frac{D}{dt}\right|_{t=0}\left(\langle\bar{p}(t),W\rangle\overline{Y}(t) - \langle\overline{Y}(t),W\rangle\bar{p}(t)\right) \\ &= \langle\overline{X},W\rangle\overline{Y} + \langle\overline{p},W\rangle\overline{Y}'(0) - \langle\overline{Y}'(0),W\rangle\overline{p} - \langle\overline{Y},W\rangle\overline{X}. \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\nabla}_X\mathcal{D}_Y W - \overline{\nabla}_Y\mathcal{D}_X W &= 2\langle\overline{X},W\rangle\overline{Y} - 2\langle\overline{Y},W\rangle\overline{X} \\ &\quad + \langle\overline{X}'(0) - \overline{Y}'(0),W\rangle\overline{p} - \langle\overline{p},W\rangle(\overline{X}'(0) - \overline{Y}'(0)) \\ &= 2\langle\overline{X},W\rangle\overline{Y} - 2\langle\overline{Y},W\rangle\overline{X} \end{aligned}$$

The last equality is justified because $[X,Y](p) = 0$ implies that $[\overline{X},\overline{Y}](\overline{p}) = 0$, which in turn implies that $\overline{X}'(0) - \overline{Y}'(0)$ is parallel to $\overline{p}$. This completes the proof. □

We end this section by mentioning a few interesting properties of the connection associated with $\{f, \lambda\}$. If $\lambda = 0$, then ∇ is clearly flat. By Lemma 5.2, if $\lambda = -2$ then ∇ is flat, which is perhaps less obvious. Regarding f as a unit-length section of the bundle, the covariant derivative of f in the direction $X \in T_pB^n$ is

$$\nabla_X f = \overline{\nabla}_X f + \lambda \mathcal{D}_X f(p) = \overline{X} + \lambda(f(p) \times \overline{X}) \times f(p) = \overline{X} + \lambda\overline{X} = (1+\lambda)\overline{X}, \tag{5.1}$$

where $\overline{X} = df_pX$. So if $\lambda = -1$, then f is a parallel section, and hence the holonomy group of ∇ is isomorphic to S^1. Thus, λ plays a significant role along with f in determining the qualitative geometric properties of ∇.

6. Rigidity with S^1-invariant Sharafutdinov fibers

In this section, we show that for a class of metrics more general than connection metrics, the inequality of Proposition 1.1 forces some rigidity for the metric at the soul. As before, M will denote the space $S^2 \times \mathbb{R}^3$ together with a metric of nonnegative curvature, Σ will denote a soul of M, and ∇ will denote the connection in the normal bundle $\nu(\Sigma)$. In the remainder of this section, we make the following assumptions:

(1) M has a curvature nullity section, i.e., a global unit-length section W of $\nu(\Sigma)$ such that $R^\nabla(\cdot,\cdot)W(p) = 0$ for all $p \in \Sigma$.
(2) The sectional curvature of a 2-plane $\sigma \subset \nu_p(\Sigma)$ depends only on the angle that σ forms with $W(p)$.

The first assumption is true in all known examples. To understand its content, define $F : \Sigma \to \mathbb{R}$ as the norm of R^∇. Let $p \in \Sigma$ be a point at which $F(p) \neq 0$. Let $X, Y \in T_p\Sigma$ be an oriented orthonormal basis. Since $R^\nabla(X,Y) : \nu_p(\Sigma) \to \nu_p(\Sigma)$ is a skew-symmetric endomorphism of a 3-dimensional vector space, there is a unit-length vector $W(p) \in \nu_p(\Sigma)$ such that $R^\nabla(X,Y)W(p) = 0$. If $F > 0$ on Σ, then a global unit-length section of $\nu(\Sigma)$, $p \mapsto W(p)$, can be constructed. In this case, there is a complimentary bundle

$$\mathbb{R}^2 \hookrightarrow W^\perp = \{U \in \nu(\Sigma) : \langle U, W\rangle = 0\} \to \Sigma.$$

Even in known examples where F is not strictly positive, it is always possible to find a curvature nullity section W, and therefore to define $W^\perp$. In general, W is not a parallel section.

For example, if ∇ is determined by $\{f : S^2 \to S^2, \lambda\}$ as in section 5, then f is a curvature nullity section, and the isomorphism class of the complimentary bundle depends on the mapping degree of f. If f is a diffeomorphism, then the complimentary bundle is isomorphic to TS^2. As a second example, if M has normal holonomy group S^1, then there is a parallel curvature nullity section. In the connection metrics of Proposition 1.2, the isomorphism class of the complimentary bundle depends on the even integer k.

The second assumption says that each fiber looks at the soul as if it admits an isometric S^1 actions with fixed direction W. This is true in the example of Section 4. If $U, V \in \nu_p(\Sigma)$ are orthonormal vectors orthogonal to $W = W(p)$, the second assumption implies $R(U,V)W = R(U,W)V = 0$. It also implies that $g_0(p) = R(W,U,U,W)$ and $g_1(p) = R(U,V,V,U)$ describe well-defined functions $g_0, g_1 : \Sigma \to \mathbb{R}$. Compare with equation 4.3, where g_0 and g_1 are constant. Proposition 1.2 implies the following relationships between these functions:

Proposition 6.1. *Let $p \in \Sigma$ and let $X \in T_p\Sigma$ be unit-length. Define $a(X) = |\nabla_X W|$, and let k_Σ denote the Gauss curvature of Σ. Then,*

$$(XF)^2 \leq k_\Sigma \left(F^2 + \frac{2}{3} hess_{g_1}(X,X) - \frac{4a(X)^2}{3}(g_1 - g_0) \right)$$
$$a(X)^2 \cdot F^2 \leq \frac{2}{3} k_\Sigma \left(hess_{g_0}(X,X) + 2a(X)^2(g_1 - g_0) \right)$$
$$0 \leq \frac{2}{3} k_\Sigma \left(hess_{g_0}(X,X) \right)$$

Proof. Let $p \in \Sigma$ and $X \in T_p\Sigma$ with $|X| = 1$. Let $V = (\nabla_X W)/a(X)$ if $a(X) \neq 0$; otherwise let $V \in \nu_p(\Sigma)$ be an arbitrary unit-length vector orthogonal to W. Let $U \in \nu_p(\Sigma)$ be such that $\{U, V, W\}$ is an orthonormal basis of W. Choose $Y \in T_p\Sigma$ so that $\{X, Y\}$ forms an oriented orthonormal basis. For $E_1, E_2 \in \nu_p(\Sigma)$, Define $G(X, E_1, E_2)$ as

$$k_\Sigma \cdot (|R^\nabla(E_1,E_2)X|^2 + (2/3)(D_{X,X}R)(E_1,E_2,E_2,E_1)) - \langle (D_X R^\nabla)(X,Y)E_1, E_2\rangle^2,$$

which is the right side minus the left side of the inequality of Proposition 1.1. The inequalities of the proposition come from: $G(X,U,V) \geq 0$, $G(X,U,W) \geq 0$ and $G(X,V,W) \geq 0$.

Notice that $|R^\nabla(W,U)X| = |R^\nabla(W,V)X| = 0$, while $|R^\nabla(U,V)X| = F$. Next, extending U and V to be parallel along the path in the direction of X, it's easy to see that:

$$\begin{aligned} \langle (D_X R^\nabla)(X,Y)W, V\rangle &= -\langle R^\nabla(X,Y)(\nabla_X W), V\rangle = 0 \\ \langle (D_X R^\nabla)(X,Y)W, U\rangle &= -\langle R^\nabla(X,Y)(\nabla_X W), U\rangle = -a(X)\cdot F \end{aligned}$$

It remains to compute $\langle (D_X R^\nabla)(X,Y)U, V\rangle$ and the three terms involving $(D_{X,X}R)$. For this we must find nice extensions of the vectors $\{U, V, W\}$ in the direction X. Let $\alpha(t)$ be the geodesic in Σ with $\alpha(0) = p$ and $\alpha'(0) = X$. Let $W(t) = W(\alpha(t))$. For a first try, let $U(t)$ and $V(t)$ be arbitrary extensions of U and V along $\alpha(t)$ so that $\{U(t), V(t), W(t)\}$ is orthonormal for all t.

By choice of $V = V(0)$, we have $W'(0) = a\cdot V(0)$, where $a = a(X)$. Also, $V'(0) = -a\cdot W + b\cdot U$ and $U'(0) = -b\cdot V$ for some b depending on the choice of extensions. We can improve our choice of extensions to achieve $b = 0$ by defining:

$$\begin{aligned} \tilde{U}(t) &= \cos(\phi(t))\cdot U(t) + \sin(\phi(t))\cdot V(t) \\ \tilde{V}(t) &= -\sin(\phi(t))\cdot U(t) + \cos(\phi(t))\cdot V(t), \end{aligned}$$

where $\phi(0) = 0$, $\phi'(0) = b$, and $\phi''(0) = \langle V''(0), U\rangle = -\langle U''(0), V\rangle$. It is straightforward to compute that $\tilde{U}'(0) = 0$, $\tilde{V}'(0) = -aW$, $\tilde{U}''(0) \perp V$, and $\tilde{V}''(0) \perp U$. To simplify notation, omit the tilde's. Using these extensions,

$$\langle (D_X R^\nabla)(X,Y)U, V\rangle = XF - \langle R^\nabla(X,Y)U'(0), V\rangle - \langle R^\nabla(X,Y)U, V'(0)\rangle = XF.$$

The proposition will follow once we verify that:

$$\begin{aligned} (D_{X,X}R)(U,V,V,U) &= \mathrm{hess}_{g_1}(X,X) - 2a^2(g_1 - g_0) \\ (D_{X,X}R)(W,U,U,W) &= \mathrm{hess}_{g_0}(X,X) + 2a^2(g_1 - g_0) \\ (D_{X,X}R)(V,W,W,V) &= \mathrm{hess}_{g_0}(X,X) \end{aligned}$$

Since the arguments are similar, we verify the third equality, leaving the first two to the reader.

$$\begin{aligned}(D_{X,X}R)(V,W,W,V) &= X((D_XR)(V,W,W,V)) - 2(D_XR)(V'(0),W,W,V)\\ &\quad -2(D_XR)(V,W'(0),W,V)\\ &= X((D_XR)(V,W,W,V)) - 0 - 0\\ &= \mathrm{hess}_{g_0}(X,X) - 2XR(V',W,W,V) - 2XR(V,W',W,V),\end{aligned}$$

Where the last two terms simplify as follows:

$$\begin{aligned}XR(V',W,W,V) &= (D_XR)(V'(0),W,W,V) + R(V''(0),W,W,V)\\ &\quad +R(V'(0),W'(0),W,V) + R(V'(0),W,W'(0),V)\\ &\quad +R(V'(0),W,W,V'(0))\\ &= 0 + \langle V''(0),V\rangle g_0 - a^2R(W,V,W,V) + 0 + 0\\ &= -a^2g_0 + a^2g_0 = 0\\ XR(V,W',W,V) &= (D_XR)(V,W'(0),W,V) + R(V'(0),W'(0),W,V)\\ &\quad +R(V(0),W''(0),W,V) + R(V,W'(0),W'(0),V)\\ &\quad +R(V,W'(0),W,V'(0))\\ &= 0 - a^2R(W,V,W,V) + \langle W''(0),W\rangle g_0 + 0 + 0\\ &= a^2g_0 - a^2g_0 = 0\end{aligned}$$

□

From the third inequality of Proposition 6.1 we immediately learn the following, with G defined as in the previous proof:

Corollary 6.2. *g_0 is a constant function, and $G(X,W,V) = 0$. Hence, the inequality of Proposition 1.1 is not quasi-strict.*

In the example of section 4, the first and second inequalities of Proposition 6.1 are both strictly satisfied. In fact, span$\{W,V\}$ is the unique plane for which $G(X,W,V) = 0$. We therefore do not expect any rigidity to follow from the first and second inequalities. However, we do get further rigidity from the fact that $G(X,\cdot,\cdot) \geq 0$ for planes very near span$\{W,V\}$. We use this idea to prove:

Lemma 6.3. *Let $\alpha(t)$ be a geodesic in Σ. Let $W(t) = W(\alpha(t))$ denote the curvature nullity section restricted to α. Then $W''(t) \in$ span$\{W(t),W'(t)\}$ for all t.*

The lemma means that, even though W is not parallel, there is a parallel 2-plane containing W along any geodesic; namely, the plane spanned by W and W'.

Proof. Let $p = \alpha(0)$ and $X = \alpha'(0)$, which we can assume is unit-length. Define $U,V \in \nu_p(\Sigma)$ as in the proof of Proposition 6.1, so that V is parallel to $W'(0)$. An arbitrary 2-plane $\sigma \subset \nu_p(\Sigma)$ is spanned by orthonormal vectors E_1, E_2 of the form:

$$\begin{aligned}E_1 &= a_1W + b_1U + c_1V = (\cos\theta)W + 0U + (\sin\theta)V\\ E_2 &= a_2W + b_2U + c_2V = (-\sin\phi\sin\theta)W + (\cos\phi)U + (\sin\phi\cos\theta)V\end{aligned}$$

Here, E_2 is an arbitrary unit-vector expressed in spherical coordinates, and E_1 is the unit-vector in the WV-plane orthogonal to E_2. Defining G as in the previous proof,

$$\begin{aligned}
&G(X,E_1,E_2)\\
&\quad= (b_1c_2-b_2c_1)^2G(X,U,V)+(a_1b_2-a_2b_1)^2G(X,W,U)\\
&\qquad -2(b_1c_2-c_1b_2)(a_1b_2-a_2b_1)\langle(D_XR^\nabla)(X,Y)U,V\rangle^2\langle(D_XR^\nabla)(X,Y)W,U\rangle^2\\
&\qquad -(4/3)b_1b_2(a_1-a_2)(c_2-c_1)(D_{X,X}R)(U,V,U,W)\\
&\qquad -(4/3)c_1c_2(a_1-a_2)(b_2-b_1)(D_{X,X}R)(U,V,W,V)\\
&\qquad -(4/3)(a_1^2c_2^2+a_2^2c_1^2-2a_1a_2c_1c_2)(D_{X,X}R)(W,V,V,W).
\end{aligned}$$

Denote $G(\phi,\theta)=G(X,E_1,E_2)$. Notice that $G(\pi/2,\theta)=G(X,W,V)=0$. It's straightforward to compute that:

$$\frac{dG}{d\phi}(\pi/2,\theta)=-(\sin\theta\cos^2\theta+\cos\theta-\cos^3\theta)\cdot(D_{X,X}R)(U,V,W,V)\geq 0.$$

Using an argument similar to the proof of Proposition 6.1, we see that:

$$(D_{X,X}R)(U,V,W,V)=(g_1-g_0)\langle W''(0),U\rangle.$$

The trigonometric expression changes sign as θ varies, so the only possibility is that $(g_1-g_0)\langle W''(0),U\rangle=0$. But we can see from Proposition 6.1 that $g_1-g_0>0$ on Σ, so $\langle W''(0),U\rangle=0$, which completes the proof. $\square$

Lemma 6.3 has a strong corollary in the special case where the connection in the normal bundle of the soul is of the type described in section 5:

Corollary 6.4. *If ∇ is induced by some diffeomorphism $f:S^2\to S^2$ and some $\lambda\in\mathbb{R}$, as described in section 5, then Σ is round and f is the identity function.*

Proof. Let $\alpha(t)$ be a geodesic in Σ, and consider $W(t)=W(\alpha(t))=f(\alpha(t))$, which we could think of as a section of $\nu(\Sigma)$ along α or as a path in $S^2(1)$. Using equation 5.1, taking one more derivative, and comparing to Lemma 6.3, we get that $f(\alpha(t))$ has no geodesic curvature. In other words, $f(\alpha(t))$ is a (possibly reparameterized) geodesic on the round sphere $S^2(1)$. Since $f:\Sigma\to S^2(1)$ maps geodesics to paths whose images are geodesics, it follows that Σ must be a round sphere of some radius, and f must be the identity map from S^2 to S^2. $\square$

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