BOUNDED RIEMANNIAN SUBMERSIONS

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ABSTRACT. In this paper, we establish global metric properties of a Riemannian submersion $\pi : M^{n+k} \to B^n$ for which the fundamental tensors are bounded in norm: $|A| \leq C_A, |T| \leq C_T$. For example, if B is compact and simply connected, then there exists a constant $C = C(B, C_A, C_T, k)$ such that for all $p \in B$, $d_{F_p} \leq C \cdot d_M$, where d_{F_p} denotes the intrinsic distance function on the fiber $F_p := \pi^{-1}(p)$, and d_M denotes the distance function of M restricted to F_p. When applied to the metric projection $\pi : M \to \Sigma$ from an open manifold of nonnegative curvature M onto its soul Σ, this property implies that the ideal boundary of M can be determined from a single fiber of the projection. As a second application, we show that there are only finitely many isomorphism types of fiber bundles among the class of Riemannian submersions whose base space and total space both satisfy fixed geometric bounds (volume from below, diameter from above, curvature from above and below).

1. INTRODUCTION

Let $\pi : M^{n+k} \to B^n$ denote a Riemannian submersion. Assume that B is compact. Let A and T denote the fundamental tensors of π. Assume that $|A| \leq C_A$ and $|T| \leq C_T$. The main purpose of this paper is to explore consequences of these bounds.

An interesting example of this situation is the metric projection $\pi : M \to \Sigma$ between an open manifold M of nonnegative sectional curvature and its soul Σ; that is, the map which sends each point $x \in M$ to the point $\pi(x)$ in Σ to which it is closest. Perelman proved in [6] that π is a well defined Riemannian submersion. The regularity of π is established in [3] to be at least C^2 and a.e. smooth. It is a trivial consequence of O'Neill's formula that the norm of the A tensor is bounded in terms of the maximal sectional curvature of Σ. Perelman established that the T tensor is bounded in terms only of the injectivity radius of the soul: $|T| \leq C_T = C_T(\text{inj}(\Sigma))$; (see [6, Theorem C]).

More generally, whenever M satisfies a (positive or negative) lower curvature bound λ, if follows from O'Neill's formula that $|A| \leq C_A = C_A(B, \lambda)$, and it follows from Perelman's argument that $|T| \leq C_T = C_T(\text{inj}(B), \lambda)$.

In section 2, we define the *holonomy group* of π. The assumed bound on the T tensor allows us to achieve certain metric bounds on the fibers of π whenever we know that the holonomy group of π is compact.

Next, in section 3, we bound the intrinsic distance function, d_{F_p}, on a fiber F_p in terms of the distance function d_M on M. More precisely, $d_{F_p} \leq C \cdot d_M$, where C depends only on B, C_A, C_T, and k.

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Next, in section 4, we apply these results to the metric projection $\pi : M \to \Sigma$ onto a soul. For example, when the soul is simply connected, we prove that the ideal boundary of M can be determined by a single fiber of π. Additionally, if π has compact holonomy, then we achieve the following splitting theorem: if the vertizonal curvatures of M decay towards zero away from the soul, then M must split locally isometrically over its soul.

Next, in section 5, we describe a very different application for the formula $d_{F_p} \leq C \cdot d_M$. Namely, this formula together with the work of P. Walczak allows us to prove that there are only finitely many isomorphism classes fiber bundles in the set of Riemannian submersions whose base space and total space both satisfy fixed geometric bounds (volume from below, diameter from above, curvature from above and below).

Finally, in section 6 we describe a general construction for bounding the "size" of the holonomy element associated to a loop in terms of the length of the loop.

In the appendix we prove that for any loop on a given compact simply connected Riemannian manifold, one can find a nulhomotopy of the loop with derivatives bounded linearly in terms of the length of the loop. This observation is central to several proofs in this paper.

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2. The holonomy group of a Riemannian submersion

For any piecewise smooth path $\alpha(t)$ in B, say from p to q, there is a naturally associated diffeomorphism h^α between the fibers $F_p := \pi^{-1}(p)$ and $F_q := \pi^{-1}(q)$. This diffeomorphism maps $x \in F_p$ to the terminal point of the lift of α to a horizontal path in M beginning at x. We will call h^α the *holonomy diffeomorphism* associated to α. If π is at least C^2 (as is the case for the metric projection onto a soul), then each fiber is a C^2 Riemannian manifold, and each holonomy diffeomorphism h^α is a C^1 map between the fibers. This regularity will be sufficient for all constructions in this paper.

Fix $p \in B$ and define the *holonomy group of* π as the group, Φ, of diffeomorphisms of the fiber F_p which occur as h^α for a piecewise smooth loop α in B at p. Φ is clearly independent of the choice of p (up to group isomorphism). Φ is not necessarily a finite dimensional Lie group. We say that π has *compact holonomy* if Φ is a compact finite dimensional Lie group.

For example, if M is a Euclidean vector bundle over B with a connection metric, then Φ is just the holonomy group of the connection, which is a (not necessarily compact) Lie subgroup of the orthogonal group. For the metric projection onto a soul, Φ is the holonomy group of the normal bundle of the soul (with it's natural connection), so again Φ is a Lie subgroup of the orthogonal group. This is because, by Perelman's Theorem, if w is a normal vector to the soul at $p = \alpha(0)$, then the horizontal lift $\bar{\alpha}(t)$ of $\alpha(t)$ beginning at $\bar{\alpha}(0) = \exp(w)$ is simply $\bar{\alpha}(t) = \exp(P_{\alpha|_{[0,t]}}(w))$, where $P_{\alpha|_{[0,t]}}(w)$ denotes the parallel transport of w along the appropriate segment of α. An example given in [10] demonstrates that Φ need not be a closed subgroup, and hence need not be compact, even when the soul is simply connected.

We next describe two consequences of compact holonomy. The first is due to Schroeder and Strake [8, Proposition 1]:

Lemma 2.1. *If Φ has compact holonomy, then there exists a constant $b_1 = b_1(\pi)$ such that $\Phi = \{h^\alpha \mid \alpha$ is a piecewise smooth loop in B at p of length $\leq b_1\}$.*

In other words, the entire holonomy group can be represented by loops in B of bounded length. Schroeder and Strake stated this only for the case of the holonomy group of the normal bundle of the soul, but their argument is valid in the above generality.

The second consequence of compact holonomy relates to the holonomy diffeomorphisms. It is straightforward to show that the holonomy diffeomorphism h^α satisfies the Lipschitz constant $e^{C_T \cdot \text{length}(\alpha)}$; see [4, Lemma 4.2]. We prove next that, in the case of compact holonomy, the holonomy diffeomorphisms satisfy a Lipschitz bound which does not depend on the length of the path. In the case of non-compact holonomy we at least find a Lipschitz bound which depends linearly, rather than exponentially, on the length of α:

Proposition 2.2.

(1) *If Φ is compact, then there exists a constant $L = L(\pi)$ such that all holonomy diffeomorphisms of π satisfy the Lipschitz constant L.*
(2) *Even if Φ is non-compact, there exists a constant $\tilde{L} = \tilde{L}(C_T, diam(B))$ such that the holonomy diffeomorphism associated with any loop of length l satisfies the Lipschitz constant $1 + l \cdot \tilde{L}$.*

Proof. For part 1, if c is a loop at p in B of arbitrary length, one can find a different loop $\tilde{c}$ of length $\leq b_1$ which generates the same holonomy diffeomorphism: $h^c = h^{\tilde{c}}$, where b_1 is the constant from Lemma 2.1. It follows that if α is a path of arbitrary length in B between points p and q, then $h^\alpha = h^{\tilde{\alpha}}$ for a properly chosen path $\tilde{\alpha}$ of length $\leq \text{diam}(B) + b_1$. For example, take $\tilde{\alpha}$ equal to a minimal path β from p to q followed by a loop at q which generates the same holonomy diffeomorphism as $\beta^{-1} * \alpha$. Therefore $L := e^{C_T(\text{diam}(B)+b_1)}$ is as required for part 1 of the proposition.

For part 2, first notice that for any loop α of length $l \leq D := 2 \cdot \text{diam}(B) + 1$, h^α satisfies the Lipschitz constant $e^{C_T \cdot l} \leq 1 + l \cdot P$, where $P := C_T \cdot e^{C_T \cdot D}$ (that is, P equals the maximum derivative of the function $e^{C_T \cdot l}$ between $l = 0$ and $l = D$). But for a unit-speed loop α at p in B of arbitrary length l, it is possible to find $\bar{l}$ loops, $\alpha_1, ..., \alpha_{\bar{l}}$, each of length $\leq D$, such that $h^\alpha = h^{\alpha_1} \circ \cdots \circ h^{\alpha_{\bar{l}}}$, where $\bar{l}$ denotes the smallest integer which is $\geq l$. To do this, define α_i to be the composition of a minimal path in B from p to $\alpha(i-1)$, followed by $\alpha|_{[i-1,i]}$, followed by a minimal path from $\alpha(i)$ to p. Since for each i, h^{α_i} satisfies the Lipschitz constant $1 + P \cdot D$, it follows that h^α satisfies the following Lipschitz constant:

$$\bar{l}(1 + P \cdot D) \leq (l+1)(1 + P \cdot D) = 1 + (1 + PD)l + PD \leq 1 + 2(1 + PD)l.$$

The final inequality holds whenever $l \geq 1$, but if $l < 1$ then h^α satisfies the Lipschitz constant $1 + (C_T \cdot e^{C_T \cdot 1})l$ (as above). So the choice $\tilde{L} := \max\{2(1 + PD), C_T \cdot e^{C_T}\}$ is as required for part 2 of the proposition. □

Guijarro and Walschap asked whether, for the metric projection onto a soul, all holonomy diffeomorphisms must satisfy a global Lipschitz bound [4]. The reason for their interest in this question will be discussed in section 4. Proposition 2.2 offers only a partial answer to this question. We do not know the full answer for the metric projection onto the soul, but for general Riemannian submersions we have the following example:

Example 2.3. For general Riemannian submersions with bounded tensors, the holonomy diffeomorphisms need not satisfy a global Lipschitz bound. For example, consider the projection $\pi : S^2 \times \mathbb{R}^2 \to S^2$, where S^2 has the round metric, $\mathbb{R}^2$ has the flat metric, and $S^2 \times \mathbb{R}^2$ has the product metric. Let X denote the vector field on S^2 whose flow is rotation about the axis through the north and south poles N, S. Let Y be a vector field on S^2 which is orthogonal to X and vanishes at N, S. Let $\frac{\partial}{\partial\theta}(v)$ $(v \in \mathbb{R}^2)$ denote the radial vector field on $\mathbb{R}^2$, and let $W(v) = \mu(|v|) \cdot \frac{\partial}{\partial\theta}(v)$, where μ is a smooth bump function with support $[1, 2]$. We can consider X, Y to be horizontal vector fields and W to be a vertical vector field on $S^2 \times \mathbb{R}^2$ in the obvious way. Define a 2-plane distribution $\mathcal{H}$ on $S^2 \times \mathbb{R}^2$ as follows:

$$\mathcal{H}(p, v) = \text{span}\left\{ Y(p), \frac{X(p)}{|X(p)|} + |X(p)| \cdot W(v) \right\}.$$

This distribution clearly extends continuously to the fibers over N, S. There is a unique metric on $S^2 \times \mathbb{R}^2$ for which π becomes a Riemannian submersion with horizontal distribution $\mathcal{H}$ such that the fibers are isometric to flat $\mathbb{R}^2$. The holonomy group Φ is isomorphic to $\mathbb{R}$, and its action on the fiber $\pi^{-1}(N) = \mathbb{R}^2$ is simply the flow along the vector field W. It is easy to see that arbitrarily long loops in S^2 are necessary to achieve arbitrarily large time parameters for this flow, and hence arbitrarily bad Lipschitz bounds for the associated holonomy diffeomorphism. But since the A and T tensors both vanish outside of a compact set in $S^2 \times \mathbb{R}^2$, they have bounded norms.

3. The intrinsic metric on the fibers

Denote $k := \dim(M) - \dim(B)$. For $p \in B$, denote by d_{F_p} the intrinsic distance function of the fiber F_p, and by d_M the distance function of M restricted to F_p. In this section we establish the following global metric property of the fibers of π:

Theorem 3.1. *If B is simply connected then:*

1. *There exists a constant $C_1 = C_1(B, C_A, C_T, k)$ such that for any $p \in B$, $d_{F_p} \leq C_1 \cdot d_M$.*
2. *If π has compact holonomy then there exists a constant $C_2 = C_2(\pi)$ such that for any two points $x, y \in F_p$ between which there exists a piecewise smooth horizontal path, $d_{F_p}(x, y) \leq C_2$.*

Notice that, although part 2 is only interesting when M is noncompact, part 1 is interesting in either case.

We begin the proof with two lemmas which provide technical bounds on a Riemannian submersion with bounded tensors. Hereafter we denote by $Y^{\mathcal{V}}$ and $Y^{\mathcal{H}}$ the vertical and horizontal components of a vector $Y \in TM$.

Lemma 3.2.

1. *Along any horizontal path $\sigma(t)$ in M one can construct an orthonormal vertical frame $\{\tilde{V}_1(t), ..., \tilde{V}_k(t)\}$ with $|\tilde{V}_i'(t)| \leq 4^k \cdot k! \cdot C_A \cdot |\sigma'(t)|$.*
2. *If $Y(t)$ is any vector field along any horizontal path $\sigma(t)$ in M, then*

$$\frac{d}{dt}|Y(t)^{\mathcal{V}}| \leq k|(Y'(t))^{\mathcal{V}}| + k \cdot 4^k \cdot k! \cdot C_A \cdot |\sigma'(t)| \cdot |Y(t)|.$$

Proof. To establish part 1, let $\sigma(t)$ be a horizontal path in M. Let $\{Y_i(t)\}$, $i = 0..k$, denote the parallel transport along $\sigma(t)$ of an orthonormal basis $\{Y_i(0)\}$ of the *vertical* space at $\sigma(0)$. Denote by $Y_i(t) = X_i(t) + V_i(t)$ the decomposition of $Y_i(t)$ into horizontal and vertical components. Notice that

$$V_i'(t) = V_i'(t)^{\mathcal{H}} + V_i'(t)^{\mathcal{V}} = V_i'(t)^{\mathcal{H}} - X_i'(t)^{\mathcal{V}} = A(\sigma'(t), V(t)) - A(\sigma'(t), X(t)).$$

Therefore, $|V_i'(t)| \leq 2C_A|\sigma'(t)|$. We next define the frame $\{\tilde{V}_i(t)\}$ as the Gram-Schmidt orthonormalization of the (ordered) frame $\{V_i(t)\}$. For example, $\tilde{V}_1(t) = \langle V_1(t), V_1(t)\rangle^{-\frac{1}{2}} V_1(t)$. Differentiating this expression gives: $|\tilde{V}_1'(0)| \leq 2|V_1'(0)| \leq 4C_A|\sigma'(0)|$. Continuing the Gram-Schmidt process gives:

$$\tilde{V}_l(t) = \left(V_l(t) - \sum_{i=1}^{l-1} \langle \tilde{V}_i(t), V_l(t)\rangle \tilde{V}_i(t) \right)_{\text{normalized}}$$

Differentiating this expression gives:

$$\begin{aligned} |\tilde{V}_l'(0)| &\leq 2\left(|V_l'(0)| + \sum_{i=1}^{l-1} (|V_l'(0)| + 2|\tilde{V}_i'(0)|) \right) \\ &\leq 2\left(2lC_A|\sigma'(0)| + 2\sum_{i=1}^{l-1} |\tilde{V}_i'(0)| \right) \leq a(l)C_A|\sigma'(0)|, \end{aligned}$$

where $a(l)$ is the solution to the following recurance relation: $a(0) = 0$; $a(l) = 4\left(l + \sum_{i=1}^{l-1} a(i)\right)$. It is easy to see that $a(l) \leq 4^l \cdot l!$, which proves part 1 of the lemma.

Part 2 of the lemma follows by writing $|Y(t)^{\mathcal{V}}|^2 = \sum_{i=1}^{k} \langle Y(t), \tilde{V}_i(t)\rangle^2$ for the frame $\{\tilde{V}_i(t)\}$ given in part 1, and then differentiating with respect to t. □

We use the previous Lemma to establish the following bound, which will be central to our proof of Theorem 3.1:

Lemma 3.3. *Let $\alpha_s(t) = \alpha(s,t)$ $(s \in [0,\epsilon), t \in [0,1])$ denote a family of piecewise-smooth paths in B with fixed endpoints: $\alpha_s(0) = p, \alpha_s(1) = q$. Assume $|\alpha_0'(t)| \leq C_1$. Assume for the variational vector field $V(t) := \frac{\partial}{\partial s}\alpha(0,t)$ along α_0 that $|V(t)| \leq C_2$. Let $x \in F_p$. For each fixed s, let $t \mapsto \bar{\alpha}_s(t) = \bar{\alpha}(s,t)$ denote the horizontal lift the path $t \mapsto \alpha_s(t)$ with $\bar{\alpha}_s(0) = x$. Then $\tau(s) := \bar{\alpha}_s(1)$ is a path in the fiber F_q, and $|\tau'(0)| \leq \rho(1)$, where $\rho(t)$ denotes the solution to the following differential equation:*

$$\rho'(t) = kC_AC_1C_2(1 + 4^k k!) + kC_1(C_T + 4^k k! C_A)\rho(t) \ ; \ \rho(0) = 0.$$

Proof. Let $\frac{\partial}{\partial t}\bar{\alpha}(s,t)$ and $\frac{\partial}{\partial s}\bar{\alpha}(s,t)$ denote the natural coordinate vector fields along the parameterized surface $\bar{\alpha}$. Notice that $\frac{\partial}{\partial t}\bar{\alpha}$ is everywhere horizontal. Also, $|\frac{\partial}{\partial t}\bar{\alpha}(0,t)| = |\frac{\partial}{\partial t}\alpha(0,t)| \leq C_1$ and $|(\frac{\partial}{\partial s}\bar{\alpha}(0,t))^{\mathcal{H}}| = |\frac{\partial}{\partial s}\alpha(0,t)| \leq C_2$.

Applying part 2 of Lemma 3.2 to the vector field $\frac{\partial}{\partial s}\bar{\alpha}(0,t)$ along the horizontal curve $t \mapsto \bar{\alpha}(0,t)$ gives:

$$\begin{aligned}
\frac{d}{dt}\Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big| &\le k\Big|\Big(\frac{\mathrm D}{\mathrm{dt}}\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big| + k4^k k! C_A C_1\Big|\frac{\partial}{\partial s}\bar\alpha(0,t)\Big| \\
&= k\Big|\Big(\frac{\mathrm D}{\mathrm{ds}}\frac{\partial}{\partial t}\bar\alpha(0,t)\Big)^{\mathcal V}\Big| + k4^k k! C_A C_1\Big|\frac{\partial}{\partial s}\bar\alpha(0,t)\Big| \\
&\le k\left|A\left(\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal H},\frac{\partial}{\partial t}\bar\alpha(0,t)\right) + T\left(\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V},\frac{\partial}{\partial t}\bar\alpha(0,t)\right)\right| \\
&\quad + k4^k k! C_A C_1\left(\Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal H}\Big| + \Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big|\right) \\
&\le kC_AC_1C_2 + kC_TC_1\Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big| \\
&\quad + k4^k k! C_A C_1\left(C_2 + \Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big|\right).
\end{aligned}$$

In other words,

$$\frac{d}{dt}\Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big| \le kC_AC_1C_2(1+4^k k!) + kC_1(C_T + 4^k k! C_A)\Big|\Big(\frac{\partial}{\partial s}\bar\alpha(0,t)\Big)^{\mathcal V}\Big|.$$

Since $\tau'(0) = \frac{\partial}{\partial s}\bar\alpha(0,1) = (\frac{\partial}{\partial s}\bar\alpha(0,1))^{\mathcal V}$, this proves the lemma. □

proof of Theorem 3.1. Let $x, y \in F_p$ and let $\gamma : [0,l] \to M$ be a shortest unit-speed path in M from x to y. Let $\bar l$ denote the smallest integer which is not less than $l := \text{length}(\gamma)$. For each integer i between 0 and $\bar l - 1$, choose a point x_i of F_p closest to $\gamma(i)$, and choose a shortest unit-speed path τ_i from x_i to $\gamma(i)$. Next, for each integer i between 1 and $\bar l$, define γ_i as the concatenation of τ_{i-1} followed by $\gamma|_{[i-1,i]}$ followed by τ_i^{-1}. By construction, γ_i is a path in M between x_{i-1} and x_i, whose length, l_i, is not greater than $D := 2\cdot\text{diam}(B)+1$. We describe next how to construct a path β_i between x_{i-1} and x_i which remains in the fiber F_p, such that the length of β_i can bounded linearly in terms of l_i.

Let $\alpha_i := \pi\circ\gamma_i$, which is a loop at p in B whose length is not greater than l_i. Let $z_i := h^{\alpha_{i+1}}(x_i)$. We first construct a path β_i^1 in F_p from x_{i-1} to z_{i-1}, and then construct a path β_i^2 in F_p from z_{i-1} to x_i.

To construct the path β_i^1, first reparameterize α_i proportional to arclength, so that $\alpha_i : [0,1] \to B$. Find a piecewise smooth nulhomotopy $H : [0,1]\times[0,1] \to B$ of α_i. That is, $H(0,t) = p, H(1,t) = \alpha_i(t)$. By Lemma 7.2 from the appendix, H can be chosen so that $|\frac{\partial}{\partial t}H| \le Ql_i$ and $|\frac{\partial}{\partial s}H| \le Q$, where Q depends only on B. Lift the homotopy H to M by defining, for each $s\in[0,1]$, the curve $t\mapsto \tilde H(s,t)$ to be the horizontal lift of the curve $t\mapsto H(s,t)$ beginning at $\tilde H(s,0) = x_{i-1}$. Let $\beta_i^1(s) := \tilde H(s,1)$. By Lemma 3.3, $|(\beta_i^1)'(s)| \le \rho_{l_i}(1)$, where $\rho_{l_i}(t)$ is the solution to the following differential equation:

$$(\rho_{l_i})'(t) = kC_AQ^2l_i(1+4^k k!) + kQl_i(C_T + 4^k k! C_A)\rho_{l_i}(t)\ ;\ \rho_{l_i}(0) = 0.$$

In particular, $\text{length}(\beta_i^1) \le C\cdot l_i$, where C is a bound on the derivative of the function $l_i \mapsto \rho_{l_i}(1)$ between $l_i = 0$ and $l = D$.

We continue by constructing a path β_i^2 in F_p between z_{i-1} and x_i, whose length is also controlled linearly in terms of l_i. Let $\beta_i^2(s) := h^s(\gamma_i(s))$, where $h^s : F_{\alpha_i(s)} \to F_p$ is the holonomy diffeomorphism associated to the curve $\alpha_i|_{[s,l_i]}$. It is clear from construction that β_i^2 connects z_{i-1} to x_i and that $(\beta_i^2)'(s) = dh^s(\gamma_i'(s)^{\mathcal V})$. Thus, length$(\beta_i^2) \leq L \cdot l_i$, where L is a Lipschitz bound on all holonomy diffeomorphisms associated to curves in B of length $\leq D$.

The concatenation, β_i, of β_i^1 followed by β_i^2 satisfies length$(\beta_i) \leq (C+L)l_i \leq (C+L)D$. The concatenation of the paths β_i $(i = 1, ..., \bar{l})$ is a path from x to y in F_p. Therefore,

$$d_{F_p}(x,y) \leq \bar{l}(C+L)D \leq (l+1)(C+L)D \leq 2l(C+L)D.$$

The final inequality is valid only when $l \geq 1$, but when $l \leq 1$, $d_{F_p}(x,y) \leq (C+L)\cdot l$. So the choice $C_1 := \max\{2(C+L)D, C+L\}$ is as required for part 1 of the theorem.

To prove part 2, let $x, y \in F_p$ be two points between which there exists horizontal path $\bar{\alpha}$. Let $\alpha := \pi \circ \bar{\alpha}$. Notice that $h^\alpha(x) = y$. One can choose a different loop σ at p in B such that $h^\sigma = h^\alpha$ and length$(\sigma) \leq b_1$, where b_1 is the constant from Lemma 2.1. The horizontal lift, $\bar{\sigma}$, of σ provides an alternative horizontal path from x to y. Thus $d_{F_p}(x,y) \leq C_1 \cdot d_M(x,y) \leq C_1 \cdot \text{length}(\sigma) \leq C_1 \cdot b_1$, so the choice $C_2 := C_1 b_1$ is as required for part 2. □

Example 3.4. The conclusions of parts 1 and 2 theorem 3.1 both fail for $\pi : TS^2 \to S^2$ when TS^2 is given the connection metric with flat fibers, which has unbounded A-tensor. It's easy to see that two rays from the origin of T_pS^2 grow apart linearly in the fiber T_pS^2, while they maintain a distance $\leq$ length(α) in TS^2, where α is a loop in S^2 which parallel translates the initial tangent vector of the first ray to the initial tangent vector of the second ray.

Example 3.5. The conclusions of parts 1 and 2 of theorem 3.1 both fail for the following example in which the base space is not simple connected. Let M be the following flat manifold: $M = \mathbb{R}^2 \times [0,1]/\{(r,\theta,0) \sim (r, \theta+\lambda, 1)\}$. Here λ is any non-zero angle. Guijarro and Petersen studied this manifold in [5]. The soul of M is the circle $0 \times [0,1]/\sim$, and each fiber F_p of the metric projection onto the soul is isometric to flat $\mathbb{R}^2$. Two rays from the same point p of the soul which make an angle λ will grow apart linearly in the fiber F_p, but will remain at distance ≤ 1 in M.

Example 3.6. In [10], a metric of nonnegative curvature on $S^2 \times \mathbb{R}^4$ is exhibited for which the holonomy group of the metric projection onto the soul is a noncompact Lie subgroup of $SO(4)$. It is easy to see that the conclusion of part 2 of Theorem 3.1 is invalid for this manifold.

4. The metric projection onto the soul

Let M be an open manifold of nonnegative sectional curvature with soul $\Sigma \subset M$. Let $\pi : M \to \Sigma$ denote the metric projection. In this section we study two applications of the previously developed theory to this Riemannian submersion. The first says that the ideal boundary of M is determined by a single fiber of π:

Corollary 4.1. *If $\pi_1(\Sigma) = 1$, then the topology of the ideal boundary of M can be determined by the pointed manifold (F_p, p) for any $p \in \Sigma$.*

Proof. Remember that the ideal boundary, $M(\infty)$, of M can be defined with reference to any fixed point $p \in M$ by declaring $M(\infty)$ to be the set of equivalence classes of unit-speed rays in M from p, endowed with the following distance function:

$$d_\infty([\gamma_1],[\gamma_2]) := \lim_{t\to\infty} \frac{1}{t} d^t(\gamma_1(t), \gamma_2(t)),$$

where d^t denotes the intrinsic distance function on the sphere of radius t about p.

Since it is difficult to work with d^t, we define an alternate metric on $M(\infty)$ as follows:

$$\tilde{d}_\infty([\gamma_1],[\gamma_2]) = \begin{cases} \infty & \text{(if } \gamma_1,\gamma_2 \text{ lie on different ends of } M\text{);} \\ \lim_{t\to\infty} \frac{1}{t} d_M(\gamma_1(t),\gamma_2(t)) & \text{(otherwise).} \end{cases} \tag{4.1}$$

By [9, Proposition 2.2], if rays γ_1 and γ_2 lie on the same end of M, then:

$$\tilde{d}_\infty([\gamma_1],[\gamma_2]) = \sqrt{2 - 2\cos(d_\infty([\gamma_1],[\gamma_2]))}.$$

It follows that d_∞ and $\tilde{d}_\infty$ induce the same topology on $M(\infty)$.

The constructions above are invariant under the choice of $p \in M$. For our purposes it is convenient to choose $p \in \Sigma$. Any ray in F_p from p is also a ray in M, so we could equally well have defined $M(\infty)$ as the set of rays in F_p from p. By Theorem 3.1, we get the same topology on $M(\infty)$ if we replace d_M with d_{F_p} in equation 4.1. The statement of the corollary follows. □

Second, we show an application of Proposition 2.2 to the metric projection onto a soul. The application is related to a splitting theorem of Guijarro and Petersen which states that if the curvatures of all 2-planes on M decay towards zero away from the soul then the soul must be flat [5]. By O'Neill's formula, one can then conclude that the A-tensor of π vanishes, and hence that M splits locally isometrically over the soul (see [4]). Guijarro and Walschap have demonstrated that if M has the property that all holonomy diffeomorphism of π obey a global Lipschitz bound, then one only needs to know that the curvatures of all vertizontal 2-planes (that is, 2-planes spanned by a horizontal and a vertical vector) decay towards zero away from the soul in order to conclude that M splits locally isometrically over its soul [4, Theorem 4.3.2]. Thus, we have as a corollary to Proposition 2.2:

Corollary 4.2. *If Φ is compact and the curvatures of vertizontal 2-planes on M decay towards zero away from Σ, then M splits locally isometrically over Σ.*

We do not know whether the statement of this corollary is necessarily true when Φ is noncompact.

5. A finiteness theorem for Riemannian submersions

In this section we prove that there are only finitely many equivalence classes of Riemannian submersions whose base space and total space both satisfy fixed geometric bounds. We consider two Riemannian submersions, $\pi_1 : M_1 \to B_1$ and $\pi_2 : M_2 \to B_2$, to be C^k-equivalent if there exists a C^k map $f : M_1 \to M_2$ which maps the fibers of π_1 to the fibers of π_2. Every Riemannian submersion is a fiber bundle, and this notion of equivalence just means equivalence up to C^k fiber bundle isomorphism.

Theorem 5.1. *Let $n, k \in \mathbb{Z}$ and $V, D, \lambda \in \mathbb{R}$. Then there are only finitely many C^1 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \to B^n$ for which:*

1. *B is simply connected.*
2. *$vol(B) \geq V, diam(B) \leq D, |sec(B)| \leq \lambda$.*
3. *$vol(M) \geq V, diam(M) \leq D, |sec(M)| \leq \lambda$.*

This theorem is based on the following result of P. Walczak ([11], as corrected in [12]):

Theorem 5.2 (Walczak). *Let $n, k \in \mathbb{Z}$ and $V, D, \lambda, C_A, C_T \in \mathbb{R}$. Then there are only finitely many C^1 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \to B^n$ for which:*

1. $|A| \leq C_A$ *and* $|T| \leq C_T$.
2. $vol(B) \geq V, diam(B) \leq D, |sec(B)| \leq \lambda$.
3. *There exists a fiber F_p for which $vol(F_p) \geq V$, $diam(F_p) \leq D$, $|sec(F_p)| \leq \lambda$.*

Proof of Theorem 5.1. Let $n, k \in \mathbb{Z}$ and let $V, D, \lambda \in \mathbb{R}$. Suppose that $\pi : M^{n+k} \to B^n$ is a Riemannian submersion satisfying conditions 1-3 of Theorem 5.1. As argued in section 1, the A and T tensors of π are bounded in norm by constants C_A and C_T depending only on λ and $\text{inj}(B)$. By a well known lemma of Cheeger, $\text{inj}(B)$ is in turn bounded below by a constant depending only on n, V, D, and λ (see [7]).

Let $p \in B$ and let $F_p := \pi^{-1}(p)$. It remains to bound the volume, diameter, and curvature of F_p in terms of $\{n, k, V, D, \lambda, C_A, C_T\}$, and then apply Theorem 5.2. First, by Gauss' formula, $|\text{sec}(F_p)| \leq \lambda + 2C_T$. Second, to control $\text{vol}(F_p)$, notice that any two fibers have similar volumes. More precisely, the diffeomorphism $h^\alpha : F_p \to F_q$ associated to a minimal path, α, in B between p and q satisfies the Lipschitz constant $e^{C_T \cdot \text{length}(\alpha)} \leq e^{C_T \cdot D}$, so $\text{vol}(F_p) \leq (e^{kC_T \cdot D}) \cdot \text{vol}(F_q)$. But by Fubini's theorem, $\text{vol}(M) = \int_{p \in B} \text{vol}(F_p) \, \text{dvol}_B$, which implies that for any $p \in B$, $\text{vol}(F_p) \geq \frac{\text{vol}(M)}{\text{vol}(B)}(e^{-kC_T \cdot D})$. By the Bishop-Gromov inequality, $\text{vol}(B)$ is bounded *above* by a constant depending only on D, λ, and n. This observation completes our argument that $\text{vol}(F_p)$ is bounded below.

Finally, $\text{diam}(F_p) \leq C_1 \cdot \text{diam}(M) \leq C_1 \cdot D$, where C_1 is the constant from Theorem 3.1, which depends on $\{B, C_A, C_T, k\}$. In fact, it is clear from the proof of theorem 3.1 that C_1 really depends only on $\{\text{diam}(B), Q(B), C_A, C_T, k\}$, where $Q(B)$ is the constant in Lemma 7.2. We argue now that $Q(B)$ depend only on the assumed geometric bounds of B. Let $\mathfrak{M}$ denote the class of all n dimensional Riemannian manifolds for which $\text{vol} \geq V$, $\text{diam} \leq D$, and $|\text{sec}| \leq \lambda$. By assumption, $B \in \mathfrak{M}$. $\mathfrak{M}$ is pre-compact in the Lipschitz topology and contains only finitely many diffeomorphism types (see [7]). This means that it is possible to choose a finite set, $\{B_1, ..., B_l\} \subset \mathfrak{M}$, and a constant $L = L(n, V, D, \lambda)$ such that for any $M \in \mathfrak{M}$, there exists an L-biLipschitz diffeomorphism between M and some B_i. Therefore, for any $M \in \mathfrak{M}$, $Q(M) \leq L^2 \cdot \max\{Q(B_i)\}$. This proves that $Q(B)$ satisfies an upper bound depending only on $\{n, V, D, \lambda\}$, which completes the proof. □

We do not know whether Theorem 5.1 is true without the hypothesis that the base space is simply connected. It is also possible that even without the upper curvature bound on the total space and/or the base space, there are still only finitely many C^0 fiber bundle isomorphism classes in this set of Riemannian submersions. Some evidence that the upper curvature bound on the total space can be eliminated in this way is provided by the following theorem of J.Y. Wu [13]:

Theorem 5.3 (J.Y. Wu). *Let B^n be a compact Riemannian manifold. Let $k \in \mathbb{Z}$ and $V, D, \lambda \in \mathbb{R}$. Assume $n \geq 4$. Then there are only finitely many*

C^0 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \to B^n$ for which:

(1) $vol(M) \geq V$, $diam(M) \leq D$, $\sec(M) \geq \lambda$.
(2) *For each* $p \in B$, $d_{F_p} = d_M$.

The advantage of Wu's theorem is that the upper curvature bound on the total space is missing. The disadvantage is that condition 2 is a very strong additional hypothesis; it is even stronger than assuming that the fibers are all totally geodesic.

6. Measuring size in the holonomy group

In this section we return to arbitrary Riemannian submersions with bounded tensors. We assume in this section that the holonomy group, Φ, is a finite dimensional Lie group, but we allow the possibility that Φ is noncompact. We develop a notion of "size" in the holonomy group, and show how to control the size of h^α in terms of length(α). This is analogous to Proposition 7.1 of the appendix.

Let α_0 denote a loop in B at p for which $h^{\alpha_0} = \text{id}$. For example, this is the case when α_0 is the trivial loop, but it may also occur for nontrivial loops. Let α_s denote a variation of α_0. Then h^{α_s} defines a path in Φ beginning at id, and $V := (h^{\alpha_s})'(0)$ is an element of the Lie algebra, $\mathcal{G}$, of Φ. Every vector of $\mathcal{G}$ can be described in this way; see Schroeder and Strake's proof of Lemma 2.1. It is natural to consider V as a vertical vector field on F_p. We write $V(x)$ for the value of this vector field at $x \in F_p$. It is clear from Lemma 3.3 that this vector field has bounded norm. This observation is particularly interesting for the metric projection onto a soul. In this setting, it has been discovered through other routes that such "holonomic" vertical vector fields are bounded; see for example [10, Proposition 4.1].

We call a left invariant metric, m, on Φ *acceptable* if the following condition is satisfied: for all $V \in \mathcal{G}$, $|V|_m \leq \sup_{x \in F_p} |V(x)|$. Notice that any left invariant metric can be made acceptable by rescaling. For $g \in \Phi$, we denote by $|g|$ the supremum over all acceptable metrics on Φ of the distance in Φ between g and id. This provides a natural notion of the "size" of a holonomy element. Notice that $|g_1 \cdot g_2| \leq |g_1| + |g_2|$. Compare the following statement to Proposition 7.1 of the appendix:

Proposition 6.1. *If B is simply connected and Φ is a finite dimensional Lie group, then there exists $C = C(B, C_A, C_T, k)$ such that for any loop α in B, $|h^\alpha| \leq C \cdot length(\alpha)$.*

Proof. By an argument used many times in this paper, it will suffice to find a constant C such that $|h^\alpha| \leq C \cdot \text{length}(\alpha)$ for all loops α of length $\leq D = 2 \cdot \text{diam}(B) + 1$. Let $\alpha : [0,1] \to B$ be a constant speed loop at $p \in B$ with $l := \text{length}(\alpha) \leq D$. By Lemma 7.2, there exists a piecewise smooth nulhomotopy $H : [0,1] \times [0,1] \to B$ of α such $|\frac{\partial}{\partial t} H| \leq Ql$ and $|\frac{\partial}{\partial s} H| \leq Q$. Here $H_0(t) = H(0,t) = p$ and $H_1(t) = H(1,t) = \alpha(t)$. Let $g(s) = h^{H_s}$, which is a piecewise smooth path in Φ between the identity and h^α. It will suffice to find C such that for any acceptable metric, m, on Φ, $|g'(s)|_m \leq C \cdot l$. So choose any fixed acceptable metric, m. For fixed $s \in [0,1]$, let σ_r be the following family of loops: $\sigma_r = H_{s+r} * (H_s)^{-1}$. Notice that $h^{\sigma_0} = \text{id}$, and $h^{H_s} \circ h^{\sigma_r} = h^{H_{s+r}}$. Let $V \in \mathcal{G}$ be the element $V := (h^{\sigma_r})'(0)$. Since the metric is left invariant, $|g'(s)|_m = |V|_m$. By Lemma 3.3, for any $x \in F_p$, $|V(x)| \leq \rho_l(1)$, where $\rho_l(t)$ denotes the solution to the following differential equation:

$$\rho_l'(t) = 2kC_AQ^2l(1+4^kk!) + 2kQl(C_T + 4^kk!C_A)\rho_l(t)\ ;\ \rho_l(0) = 0.$$

In particular, $|g'(s)|_m \leq C \cdot l$, where C is a bound on the derivative of the function $l \mapsto \rho_l(1)$ between $l = 0$ and $l = D$. This completes the proof. □

We remark that Proposition 6.1 implies a weak version of Proposition 7.1. More precisely, if the unit sphere bundle of a Riemannian vector bundle with a connection is endowed with the natural connection metric, then the projection map becomes a Riemannian submersion. The T tensor of this submersion vanishes, and the A tensor depends on R^∇. This argument produces only a week version of Proposition 7.1, because it does not establish the bound to be linear in C_R.

7. Appendix

In this section we show that, in a vector bundle, the "size" of the holonomy element associated to a loop can be controlled linearly in terms of the length of the loop. This result provides motivation for Proposition 6.1. Let $\mathbb{R}^k \to E \to B$ be an Riemannian vector bundle. Assume that B is compact and simply connected. Let ∇ be a connection which is compatible with the inner products on the fibers. Let R^∇ denote the curvature tensor of ∇, and assume that $|R^\nabla| \leq C_R$. Fix $p \in B$. Let Φ denote the holonomy group, and $\mathcal{G}$ its Lie algebra. Notice that Φ is a Lie subgroup of the orthogonal group, since it acts naturally by isometries on the unit sphere E_p^1 of the fiber E_p at p. For $V \in \mathcal{G}$ and $w \in E_p^1$, denote by $V(w)$ the value at w of the vector field on E_p^1 associated to V. We call a left-invariant metric, m, on Φ *acceptable* if for all $V \in \mathcal{G}$, $|V|_m \leq \sup_{w\in E_p^1}|V(w)|$. Notice that any left invariant metric on Φ can be made acceptable by rescaling. For $g \in \Phi$, we define $|g|$ as the supremum over all acceptable metrics on Φ of the distance between g and id. This provides a natural notion of "size" in the holonomy group.

Proposition 7.1. *There is a constant $C = C(B)$ such that for any piecewise smooth loop α in B, $|P_\alpha| \leq C \cdot C_R \cdot length(\alpha)$.*

The proof of this proposition turns on the following lemma, which is also invoked in the proof of Theorem 3.1:

Lemma 7.2. *Let B be a compact and simply connected Riemannian manifold. There exists a constant $Q = Q(B)$ such that for any piecewise smooth loop $\alpha : [0,1] \to B$ (parameterized proportional to arclength), there exists a piecewise smooth nulhomotopy $H : [0,1] \times [0,1] \to B$ of α (that is, $H(0,t) = p = \alpha(0), H(1,t) = \alpha(t)$) for which the natural coordinate vector fields along the image of H are everywhere bounded in norm as follows: $|\frac{\partial}{\partial s}H| \leq Q$ and $|\frac{\partial}{\partial t}H| \leq Q \cdot length(\alpha)$. In particular, this implies that the area, $A(H)$, of the image of H is $\leq Q^2 \cdot length(\alpha)$.*

Proof. Let $\alpha : [0,1] \to B$ be a piecewise smooth loop in B. Assume that α is parameterized proportional to arclength. Denote $p := \alpha(0) = \alpha(1)$, and $l :=$ length(α). To start, we will assume that $l \leq \frac{1}{2}\text{inj}(B)$, in which case it is easy to construct a nulhomotopy of α with derivative information controlled linearly in terms of l. Since $l \leq \frac{1}{2}\text{inj}(B)$, α lifts to a loop $\tilde{\alpha} = \exp_p^{-1} \circ \alpha$ at 0 in T_pB. The natural nulhomotopy of $\tilde{\alpha}$ is $\tilde{H}(s,t) := s \cdot \tilde{\alpha}(t)$ $(s \in [0,1], t \in [0,1])$. Clearly

$|\frac{\partial}{\partial t}\tilde{H}|, |\frac{\partial}{\partial s}\tilde{H}| \leq l$. Letting $H := \exp_p \circ \tilde{H}$, which is a piecewise smooth nulhomotopy of α, we see that $|\frac{\partial}{\partial t}H|, |\frac{\partial}{\partial s}H| \leq Q_1 \cdot l$ for a properly chosen $Q_1 = Q_1(B)$.

Next we assume only that $l \leq D := 2 \cdot \text{diam}(B) + 1$. More precisely, we seek a constant Q_2 such that for any piecewise smooth loop α in B at p of length $\leq D$, there exists a piecewise smooth nulhomotopy H of α, with $|\frac{\partial}{\partial t}H|, |\frac{\partial}{\partial s}H| \leq Q_2$.

Suppose no such Q_2 exists. Then there must be a sequence $\alpha_i : [0,1] \to B$ of piecewise smooth loops, each with length $\leq D$, such that the minimal derivative bounds of piecewise smooth nulhomotopies of the loops α_i go to infinity. By restricting to a subsequence, we can assume that α_i converges in the sup norm; in particular α_i is a Cauchy sequence. For i, j large enough that $\epsilon := d_{\text{sup}}(\alpha_i, \alpha_j) \leq \frac{1}{2}\text{inj}(B)$, the natural piecewise smooth homotopy $H : [0,\epsilon] \times [0,1] \to B$ between $H_0(t) = H(0,t) = \alpha_i(t)$ and $H_\epsilon(t) = H(\epsilon, t) = \alpha_j(t)$, which retracts along shortest geodesics between corresponding points of the two curves, is well defined. More precisely, define $H(s,t) = c_t(s)$, where c_t is the minimal path between $c_t(0) = \alpha_i(t)$ and $c_t(\epsilon) = \alpha_j(t)$, parameterized so as to have the constant speed $d(\alpha_i(t), \alpha_j(t))/\epsilon$. Clearly $|\frac{\partial}{\partial s}H| \leq 1$. Further, $|\frac{\partial}{\partial t}H| \leq K$ for an appropriate constant $K = K(B)$. To see this, notice that for fixed t, the vector field $J_t(s) = \frac{\partial}{\partial t}H(s,t)$ along the geodesic $c_t(s)$ is a Jacobi field because it is the variational vector field of the family of geodesics which defines the homotopy. $J_t(s)$ is determined by its endpoints $V_1 := J_t(0) = \alpha_i'(t)$ and $V_2 := J_t(\epsilon) = \alpha_j'(t)$, each of whose norm is $\leq D$. We can thus take K as the supremum (over all pairs of vectors $V_1 \in T_{p_1}B, V_2 \in T_{p_2}B$ with $d(p_1,p_2) \leq \frac{1}{2}\text{inj}(B)$ and $|V_1|, |V_2| \leq D$) of the maximal norm of the Jacobi field along the shortest geodesic between p_1 and p_2 with end values V_1 and V_2.

It follows that any piecewise smooth nulhomotopy H_i of α_i can be extended to a piecewise smooth nulhomotopy H_j of α_j, with similar derivative bounds. More precisely, if $|\frac{\partial}{\partial s}H_i| \leq Q_s$ and $|\frac{\partial}{\partial t}H_i| \leq Q_t$, then $|\frac{\partial}{\partial s}H_j| \leq Q_s + E(\epsilon)$ and $|\frac{\partial}{\partial t}H_j| \leq \max\{Q_t, K\}$, where $\lim_{\epsilon\to 0} E(\epsilon) = 0$. This provides a contradiction. Therefore, such a constant Q_2 exists.

Finally, we handle the case where $l = \text{length}(\alpha)$ is arbitrary. It is possible to find $\bar{l}$ loops, $\alpha_1, ..., \alpha_{\bar{l}}$, each of length $\leq D$, such that $P_\alpha = P_{\alpha_1} \circ \cdots \circ P_{\alpha_{\bar{l}}}$, where $\bar{l}$ denotes the smallest integer which is $\geq l$. This is done exactly as in the proof of Lemma 2.2, by defining α_i to be the composition of a minimal path in B from p to $\alpha(i-1)$, followed by $\alpha|_{[i-1,i]}$, followed by a minimal path from $\alpha(i)$ to p. Let $\gamma : [0,1] \to B$ denote the composition of the loops α_i, re-parameterized proportional to arclength. Notice that $\text{length}(\gamma) \leq \bar{l} \cdot D$. It is straightforward to define a piecewise smooth homotopy $H : [0, \frac{1}{2}] \times [0,1] \to B$ between $H(0,t) = \alpha(t)$ and $H(\frac{1}{2}, t) = \gamma(t)$ with $|\frac{\partial}{\partial s}H| \leq 2 \cdot \text{diam}(B)$ and $|\frac{\partial}{\partial t}H| \leq \bar{l} \cdot D$. Next extend H by defining $H : [\frac{1}{2}, 1] \times [0,1] \to B$ as the nulhomotopy of γ which simultaneously performs nulhomotopies of each loop α_i. H is clearly a piecewise smooth nulhomotopy of α for which $|\frac{\partial}{\partial s}H| \leq \max\{2 \cdot \text{diam}(B), 2Q_2\}$ and $|\frac{\partial}{\partial t}H| \leq \bar{l}Q_2 \leq (l+1)Q_2 \leq 2lQ_2$.

The final inequality above assumes that $l \geq 1$, but the case $l \leq 1$ can be handled as follows: If $l \leq \frac{1}{2}\text{inj}(B)$ then there exists a homotopy with $|\frac{\partial}{\partial s}H| \leq Q_1 \cdot l \leq Q_1$ and $|\frac{\partial}{\partial t}H| \leq Q_1 \cdot l$. On the other hand, if $\frac{1}{2}\text{inj}(B) \leq l \leq 1$ then there exists a homotopy with $|\frac{\partial}{\partial s}H| \leq \tilde{Q}_2$ and $|\frac{\partial}{\partial t}H| \leq \tilde{Q}_2 = \frac{\tilde{Q}_2 l}{l} \leq \frac{2\tilde{Q}_2}{inj(B)}l$. Here $\tilde{Q}_2$ is derived similarly to Q_2, but for loops of length ≤ 1 rather than loops of length $\leq D$. In all cases, $|\frac{\partial}{\partial t}H|$ is bounded linearly in terms of l, and $|\frac{\partial}{\partial s}|$ is bounded absolutely, which completes the proof. □

Next we prove Proposition 7.1

Proof. Let $\alpha : [0,1] \to B$ be a unit-speed piecewise smooth loop in B at p. By Lemma 7.2, there exists a piecewise smooth nulhomotopy $H : [0,1]\times[0,1] \to B$ of α such whose area $A(H)$ is bounded linearly in terms of length(α). We now describe how to control $|P_\alpha|$ linearly in terms of $A(H)$. Let $g(s) = P_{\{t\to H_s(t)\}}$, which is a piecewise smooth path in Φ between the identity and P_α. For any vector w in E^1_p, we define $w(s,t)$ as the parallel transport of w along $t \mapsto H_s(t)$. Then,

$$\begin{aligned}
|P_\alpha| &\leq \int_0^1 \sup\{|g'(s)|_m : m \text{ is acceptable}\}\,\mathrm{ds} \\
&\leq \int_0^1 \sup_{w\in E^1_p} \left|\frac{D}{dS}\Big|_{S=s} g(S)(w)\right| \mathrm{ds} \\
&= \int_0^1 \sup_{w\in E^1_p} \left|\frac{D}{dS}\Big|_{S=s} w(S,l)\right| \mathrm{ds} \\
&\leq \int_0^1 \sup_{w\in E^1_p} \int_0^1 \left|\frac{D}{dt}\frac{D}{ds} w(s,t)\right| \mathrm{dt}\,\mathrm{ds} \\
&= \int_0^1 \sup_{w\in E^1_p} \int_0^1 \left|R^\nabla\left(\frac{\partial}{\partial t}H(s,t), \frac{\partial}{\partial s}H(s,t)\right) w(s,t)\right| \mathrm{dt}\,\mathrm{ds} \\
&\leq \int_0^1 \int_0^1 \sup_{w\in E^1_p} \left|R^\nabla\left(\frac{\partial}{\partial t}H(s,t), \frac{\partial}{\partial s}H(s,t)\right) w(s,t)\right| \mathrm{dt}\,\mathrm{ds} \\
&\leq \left|R^\nabla\right| \int_0^1 \int_0^1 \left|\frac{\partial}{\partial s}H(s,t) \wedge \frac{\partial}{\partial t}H(s,t)\right| \mathrm{dt}\,\mathrm{ds} \\
&= C_R A(H).
\end{aligned}$$

This completes the proof. ∎

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