

THE GEOMETRY OF OPEN MANIFOLDS OF NONNEGATIVE CURVATURE

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ABSTRACT
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In this paper we study the global geometric properties of an open manifold with nonnegative sectional curvature. Cheeger and Gromoll's well-known Soul Theorem states that any such manifold, M , contains a compact totally geodesic submanifold, $\Sigma \subset M$, called the "soul of M ", whose normal bundle is diffeomorphic to M . In 1994, Perelman proved that the metric projection $\pi : M \rightarrow \Sigma$ is a well defined Riemannian submersion. Perelman's breakthrough greatly simplifies the study of this class of manifolds. The main purpose of this paper is to explore consequences of Perelman's result. Along the way we develop some general theory for Riemannian submersions which is of interest independent of its application to nonnegative curvature.

For example, we derive basic properties of Riemannian submersions with compact holonomy. For the metric projection onto a soul, we show by example that the holonomy is not necessarily compact, even when the soul is simply connected, and we establish certain rigidity when the holonomy is compact.

Additionally, we develop some general theory for "bounded Riemannian submersion" (submersions whose A and T tensors are both bounded in norm). When applied to the metric projection onto a soul, $\pi : M \rightarrow \Sigma$, this theory implies that M is quasi-isometric to any single fiber of π . Perelman's theorem also enables us to bound the volume growth of M from above and below.

Finally, we study the converse of the Soul Theorem; that is, the question of which vector bundles over spheres (or more general souls) admit metrics of nonnegative curvature. For example, we prove that only finitely many vector bundles over a given soul admit a nonnegatively curved metric satisfying a fixed upper bound for the vertical curvatures at the soul. Also, we translate the question of whether a bundle admits nonnegative curvature into the question of whether it admits a connection and a tensor which together satisfy a certain differential equation.

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Chapter 1

Introduction

In this paper we study the global geometric properties of an open manifold with nonnegative sectional curvature. Cheeger and Gromoll's well-known Soul Theorem states that any such manifold, M , contains a compact totally geodesic submanifold, $\Sigma \subset M$, called the “soul of M ”, whose normal bundle, $\nu(\Sigma)$, is diffeomorphic to M . In 1994, Perelman proved that the metric projection $\pi : M \rightarrow \Sigma$ is a well defined Riemannian submersion. Perelman's breakthrough greatly simplifies the study of this class of manifolds. The main purpose of this paper is to explore consequences of Perelman's result. Along the way we develop some general theory for Riemannian submersions which is of interest independent of its application to nonnegative curvature.

Chapter 2: Background. In this chapter, we review the Soul Theorem, Perelman's Theorem, and the definition and basic properties of Riemannian submersions. Additionally, we review the definition and basic properties of two important geometric objects which, together with the soul, helps one study M ; namely, the ideal boundary, $M(\infty)$, of M , and the holonomy group, Φ , of the normal bundle, $\nu(\Sigma)$, of Σ .

Chapter 3: Conditions for nonnegative curvature on vector bundles. In the first section of this chapter, we derive a formula for the curvature of an arbitrary 2-plane at the soul. In the second section, we study the curvature of an arbitrary 2-plane “near” the soul. More precisely, imagine a family $\sigma(t)$ of 2-planes on M beginning with a mixed 2-plane $\sigma(0)$ at a point of Σ (by a mixed 2-plane, we mean a plane spanned by a tangent vector and a normal vector to Σ ; such a 2-plane necessarily has zero curvature) and then varying in such a way that the base point moves away from Σ while simultaneously the plane twists away from its initial “mixed” configuration. We study the derivatives of the curvature function $t \mapsto K(\sigma(t))$.

In the remaining sections of this chapter, we use these curvature formulas to study conditions under which a vector bundle admits a metric of nonnegative curvature. More precisely, we translate the question of whether a vector bundle admits a metric of nonnegative curvature into the question of whether it admits a connection and a tensor which together satisfy a certain differential equation. This generalizes the work of Walschap and Strake on the question of which vector bundles admit connection metrics of nonnegative curvature.

Chapter 4: Riemannian submersions with compact holonomy. We say that a Riemannian submersion $\pi : M \rightarrow B$ has compact holonomy if its holonomy group is a compact finite dimensional Lie group. For the metric projection onto a soul, $\pi : M^{n+k} \rightarrow \Sigma^n$, the holonomy group of π is just the holonomy group of the normal bundle of Σ , and is therefore a Lie subgroup of the orthogonal group $O(k)$. We show by example that it need not be a compact Lie subgroup of $O(k)$, even when the soul is simply connected. When the metric projection onto a soul does happen to have compact holonomy, we achieve the following splitting theorem: if the vertical curvatures of M decay towards zero away from Σ , then M splits locally isometrically over Σ . This theorem is based on the work of Guijarro and Walschap, and generalizes a result of Guijarro and Petersen.

Chapter 5: Volume growth bounds. In this chapter we use Perelman’s Theorem to study the volume growth, $\text{VG}(M)$, of M . In particular, $\text{VG}(M)$ is proven to be bounded above by the difference between the codimension of the soul and the maximal dimension of an orbit of the action of the holonomy group Φ on a fiber $\nu_p(\Sigma)$. Additionally, if M satisfies an upper curvature bound, then we prove that $\text{VG}(M)$ is bounded below by one plus the dimension of the ideal boundary, $M(\infty)$, of M .

Chapter 6: Bounded Riemannian submersions. In this chapter we generalize away from nonnegative curvature in order to study Riemannian submersions whose A and T tensors are both bounded in norm. Suppose that $\pi : M^{n+k} \rightarrow B^n$ is a Riemannian submersion, B is compact and simply connected, $|A| \leq C_A$, and $|T| \leq C_T$. Our main theorem provides a bound on the “folding of the fibers” of π . Specifically, if d_{F_p} denotes the intrinsic distance function on a fiber $F_p = \pi^{-1}(p)$, and d_M denotes the distance function of M restricted to F_p , then $d_{F_p} \leq C \cdot d_M$, where C depends on B , C_A , C_T , and k .

Since, by Perelman’s Theorem, the metric projection onto a soul is a bounded Riemannian submersion, we have the following application to nonnegative curvature: when the soul is simply connected, the ideal boundary of M can be determined completely from a single fiber of π .

We also explore consequences of our “folding of the fibers” bound outside of the field of nonnegative curvature; in particular, we prove that there are only finitely many isomorphism types among the class of Riemannian submersions whose base space and total spaces both satisfy fixed geometric bounds. In particular, we generalize a finiteness theorem of J.Y. Wu.

Chapter 7: Finiteness theorems for fiber bundles. In this chapter we prove finiteness theorems for vector bundles and principal bundles. For example, there are only finitely many vector bundles of a fixed rank over a fixed compact Riemannian manifold capable of admitting a connection whose curvature tensor satisfies a fixed bound in norm. An analogous statement holds for principal bundles, and the proof of this analog is based on the “folding of the fibers” bound in Chapter 6.

We provide two applications of these finiteness theorems to nonnegative curvature. First, there are only finitely many vector bundles of a fixed rank over a fixed soul (that is, a fixed compact Riemannian manifold of nonnegative curvature) which are capable of admitting nonnegatively curved metrics for which the vertical curvatures at the soul satisfy a fixed upper bound. Second, we bound the number of rank k vector bundles over a fixed Bieberbach manifold B which can admit a metric of nonnegative curvature (the bound is in terms of k and the topology of B).

Chapter 2

Background

In this chapter, we review the Soul Theorem, Perelman’s Theorem, and the definition and basic properties of Riemannian submersions. Additionally, we review the definition and basic properties of two important geometric objects which, together with the soul, helps one study M ; namely, the ideal boundary, $M(\infty)$, of M , and the holonomy group, Φ , of the normal bundle, $\nu(\Sigma)$, of Σ .

2.1 The metric projection onto a soul

Our primary object of study in this paper is an open Riemannian manifold, M , with nonnegative sectional curvature. The term “open” means complete and noncompact. One might be lead to suspect rigidity among this class of manifolds by considering the following two facts:

- Any open Riemannian manifold must contain at least one ray.
- A Riemannian manifold with sectional curvature bounded below by $\delta > 0$ is necessarily compact, and hence cannot contain any rays. Analogously, a Riemannian manifold with nonnegative sectional curvature might be thought of as “wanting to close up”, or “barely able to contain a ray”.

In 1972, Cheeger and Gromoll proved their celebrated “Soul Theorem”. By harnessing this tension between the necessity that such manifolds contain rays and their tendency to close up, Cheeger and Gromoll proved the following remarkable structure theorem [6]:

Theorem 2.1.1 (Cheeger and Gromoll’s Soul Theorem). *Any open manifold, M , with nonnegative sectional curvature contains a compact and totally convex submanifold, $\Sigma \subset M$ (called the “soul of M ”), whose normal bundle is diffeomorphic to M .*

Cheeger and Gromoll conjectured that there should exist a Riemannian submersion from M onto its soul (the reader unfamiliar with Riemannian submersions should jump ahead to Section 2.4). The resolution of this conjecture by Perelman in 1994 was a crucial breakthrough in the study of nonnegatively curved open manifolds. It provides a powerful description of the ways in which the geometries of the manifold and its soul interrelate. We believe that Perelman’s result opens the field to a renewed investigation. The main purpose of our paper is to explore consequences of Perelman’s Theorem. Perelman’s Theorem is based on the following result due to Sharafutdinov [28] (a nice alternative proof due to Croke, Schroeder and Yim is found in [44]):

Theorem 2.1.2 (Sharafutdinov). *There exists a distance non-increasing map $\pi : M \rightarrow \Sigma$.*

Since Riemannian submersions are distance non-increasing, π was a good candidate for the conjectured Riemannian submersion. However, Sharafutdinov constructed π as a limit of an infinite

sequence of rather complicated maps, which made it very difficult to study. Perelman overcame this difficulty by providing a very simple description of π , while simultaneously proving that π is in fact a Riemannian submersion.

In order to state Perelman's Theorem, we establish the following notation. Let $\Pi : \nu(\Sigma) \rightarrow \Sigma$ denote the normal bundle of Σ in M . Let $\exp^\perp : \nu(\Sigma) \rightarrow M$ denote the normal exponential map. For $p \in \Sigma$, let $\nu_p(\Sigma) := \Pi^{-1}(p)$. If γ is a path in Σ with $\gamma(0) = p$ and $V \in \nu_p(\Sigma)$, let $P_\gamma(V)$ denote the parallel transport of V along γ . Perelman's Theorem states [24]:

Theorem 2.1.3 (Perelman).

1. *For any $p \in \Sigma$, $X \in T_p\Sigma$, and $V \in \nu_p(\Sigma)$, the surface $(s, t) \rightarrow \exp^\perp(s \cdot V(t))$ ($s, t \in \mathbb{R}, s \geq 0$), where $V(t)$ denotes the parallel transport of V along the geodesic with initial tangent vector X , is a flat and totally geodesic half plane (we will refer to these surfaces as “Perelman flats”).*
2. $\pi \circ \exp^\perp = \Pi$.
3. π is a C^1 Riemannian submersion. The horizontal distribution of π is the image under $d\exp^\perp$ of the distribution in $\nu_p(\Sigma)$ which is associated to the natural connection. Said differently, if $\gamma(t)$ is a path in Σ , $p = \gamma(0)$, $V \in \nu_p(\Sigma)$, and $V(t) := P_{\gamma|_{[0,t]}}(V)$, then the path $t \mapsto \exp^\perp(V(t))$ is horizontal with respect to π .
4. The second fundamental forms of the fibers of π are globally bounded in norm.

Some comments about Perelman's Theorem are in order:

- Perelman's proof makes no mention of Sharafutdinov's intricate construction of π ; it uses only the property that π is distance non-increasing. Thus, π is the unique distance non-increasing map from M onto Σ .
- Part 2 of Perelman's Theorem says that π is simply the inverse of the normal exponential map of the soul. Said differently, π is the “metric projection”; that is, the map which sends a point $x \in M$ to the point $\pi(x) \in \Sigma$ to which it is closest. Accordingly, we will refer to π either as the “Sharafutdinov map” or as the “metric projection onto the soul”.
- Perelman's Theorem does NOT imply that $\exp^\perp : \nu(\Sigma) \rightarrow M$ is injective; however, if two different vectors of $\nu(\Sigma)$ have the same image under \exp^\perp , then the two vectors must have the same base point $p \in \Sigma$.
- Luis Guijarro has recently proven that π is at least C^2 and a.e. smooth [13].

Part 4 of Perelman's result is of crucial importance for our paper, and we wish to elaborate on it here. Perelman's precise claim was that the eigenvalues of the second fundamental forms of the fibers of π are bounded above, in barrier sense, by the reciprocal of the injectivity radius of Σ . Since Guijarro has established that π is C^2 , the second fundamental form of the fibers is well defined by the usual equation and is continuous, which frees us to ignore the phrase “in barrier sense”. Perelman's proof of this bound actually proves the following more general fact:

Proposition 2.1.4 (Generalization of Perelman's Bound). *Suppose that $f : M \rightarrow B$ is any C^2 Riemannian submersion between Riemannian manifolds. If the curvature of M is bounded below by k and the injectivity radius of B is bounded below by $i > 0$, then the second fundamental forms of the fibers of f are bounded in norm by a constant which depends only on λ and i .*

Since Perelman's proof of this bound is very abbreviated (it is a single sentence), we choose to include an elaboration of his proof here.

Proof of Proposition 2.1.4. Let $p \in B$ and let $\bar{p} \in F_p := \pi^{-1}(p)$. Let X be any unit-length vector which is normal to F_p at \bar{p} . Let V be any unit-length tangent vector to F_p at \bar{p} which is an eigenvector of the shape operator \mathbb{S}_X of F_p in the direction X ; that is $\mathbb{S}_X V = \lambda V$. We wish to bound λ .

Take any real number $r < i$. Let $\bar{\alpha} : [0, r] \rightarrow M$ denote the horizontal geodesic with $\bar{\alpha}'(0) = X$. Let $\bar{q} := \bar{\alpha}(r)$. $\bar{\alpha}$ is distance-minimizing, and in fact it continues to minimize for a short time beyond \bar{p} simply because the same is true of its projection $\alpha := \pi \circ \bar{\alpha}$. Therefore the distance sphere S about \bar{q} of radius r is smooth at \bar{p} . Also, S intersects F_p only at \bar{p} ; for, if there were another point x of intersection, then the projection via π of a minimal path from \bar{q} to x would provide a minimal path from $q := \pi(\bar{q})$ to p which is different than α , contradicting the injectivity radius bound.

The idea is to compare the second fundamental form of F_p at \bar{p} to the second fundamental form of S at \bar{p} . To assist us in doing so, we reduce to a 2-dimensional picture. Let W denote the exponential image of a small ball in the 2-plane at \bar{p} spanned by X and V . W clearly meets both S and F_p transversally. Let $\gamma : (-\epsilon, \epsilon) \rightarrow W$ denote the unit-speed parameterization of the curve of intersection of F_p and W , with $\gamma(0) = \bar{p}$. Notice that

$$\lambda = \langle \mathbb{S}_X V, V \rangle = \langle \ddot{\gamma}(0), X \rangle,$$

which is just the curvature at \bar{p} of γ .

Similarly, if \tilde{S} denotes the shape operator of S , then $\tilde{S}_X V = \langle \ddot{\beta}(0), X \rangle$, where β is the curve along which W meets S . By standard comparison theory, \tilde{S} has norm bounded in terms of r and k . Finally, it is straightforward to show that $\langle \ddot{\gamma}(0), X \rangle \leq \langle \ddot{\beta}(0), X \rangle$ simply because γ and β meet only at \bar{p} , and β is the “inner” of the two curves with respect to the orientation of W given by the vector X . \square

We end this section by mentioning an important consequence of Perelman’s Theorem which was observed by Guijarro and Walschap in [17, Proposition 2.4] (the result is also mentioned in [12, Lemma 2.2]):

Proposition 2.1.5 (Guijarro-Walschap). *For sufficiently small $r > 0$, the closed ball $\bar{B}_\Sigma(r) = \{\bar{p} \in M \mid d(\bar{p}, \Sigma) \leq r\}$ is convex.*

Since we will later need to generalize this result, we include Guijarro and Walschap’s proof of this proposition.

Proof. It will suffice to prove that the second fundamental form Π of the boundary $S_\Sigma(r)$ of $\bar{B}_\Sigma(r)$ associated to the inward-pointing unit normal vector field N is positive semidefinite. Let $\bar{p} \in S_\Sigma(r)$, let $p := \pi(\bar{p}) \in \Sigma$, and let $\gamma : [0, r] \rightarrow M$ be a minimal geodesic from $\gamma(0) = p$ to $\gamma(r) = \bar{p}$. Clearly $N(\bar{p}) = -\gamma'(r)$. Let \mathbb{S}_N denote the shape operator of $S_\Sigma(r)$ associated to N .

By Perelman’s Theorem, N is parallel along any horizontal geodesic. Hence for any $X \in \mathcal{H}_{\bar{p}}$, $\Pi(X, *) = \langle \mathbb{S}_N(X), * \rangle = 0$.

On the other hand, the set $\nu_p(\Sigma, r) := \{W \in \nu_p(\Sigma) \mid |W| = r\}$ satisfies:

$$\exp^\perp(\nu_p(\Sigma, r)) = S_\Sigma(r) \cap \partial B_p(r).$$

This set is smooth for small $r > 0$, and its tangent space at \bar{p} is $\mathcal{V}_{\bar{p}}^\perp := \{V \in \mathcal{V}_{\bar{p}} \mid V \perp N_{\bar{p}}\}$. Since balls of small radius are strictly convex, $\Pi(V, V) > 0$ for any nonzero $V \in \mathcal{V}_{\bar{p}}^\perp$. Hence, the second fundamental form of $S_\Sigma(r)$ is positive semidefinite for sufficiently small r . \square

2.2 The normal holonomy group of a soul

The holonomy group, Φ , of $\nu(\Sigma)$ plays an important role in the study of M . We remind the reader that Φ is defined with respect to a fixed $p \in \Sigma$ as the group of all orthogonal endomorphisms of

$\nu_p(\Sigma)$ which occur as P_α for some piecewise smooth loop α in Σ at p . Φ is a Lie subgroup of the orthogonal group $O(\nu_p(\Sigma))$. The identity component, Φ^0 , of Φ can be shown to be the subgroup of all endomorphism which occur as P_α for some nullhomotopic loop α in Σ at p . Φ^0 is often called the “reduced holonomy group”.

Part 3 of Perelman’s Theorem says that the connection in $\nu(\Sigma)$ controls the horizontal distribution of π . Therefore, Perelman’s Theorem implies that, in certain ways, the natural action of Φ on $\nu_p(\Sigma)$ controls the geometry of M . This will be an important theme of our paper. A well known result in this vein is the following splitting theorem:

Theorem 2.2.1 (The Splitting Theorem).

1. If $\Phi^0 = id$, then M splits locally isometrically over Σ .
2. If $\Phi = id$, then M splits globally isometrically over Σ .

“ M splits locally isometrically over Σ ”, means that for any contractible neighborhood U in Σ , $\pi^{-1}(U)$ is isometric to the Riemannian product $U \times (\mathbb{R}^k, g)$, where g is some metric of nonnegative curvature on \mathbb{R}^k . “ M splits globally isometrically over Σ ”, means that M is isometric to $\Sigma \times (\mathbb{R}^k, g)$.

Part 2 of this theorem was proven independently by Strake [33], Yim [45] and Marenich [20]. Part 1 does not seem to be explicitly stated in the literature, but it is well-known. We will mention in Remark 3.1.2 an elementary proof of part 1 which is made possible by Perelman’s Theorem together with a result of Walschap.

2.3 The ideal boundary

Together with the soul, Σ , and the holonomy group, Φ , a third important space needed to study M is the “ideal boundary of M ”, denoted $M(\infty)$. $M(\infty)$ is a compact space which encodes information about the geometry of M “at infinity”, or infinitely far away from a fixed soul. In this section we will review the definition of $M(\infty)$.

$M(\infty)$ can be defined by choosing any fixed point $p \in M$ and declaring $M(\infty)$ to be the set of equivalence classes of unit-speed rays in M from p , endowed with the following distance function:

$$d_\infty([\gamma_1], [\gamma_2]) := \lim_{t \rightarrow \infty} \frac{1}{t} d^t(\gamma_1(t), \gamma_2(t)),$$

where d^t denotes the intrinsic distance function on the sphere of radius t about p . Two rays are considered equivalent if they have distance zero.

Since it is difficult to work with d^t , we define an alternate metric on $M(\infty)$ as follows:

$$\tilde{d}_\infty([\gamma_1], [\gamma_2]) = \begin{cases} \infty & \text{(if } \gamma_1, \gamma_2 \text{ lie on different ends of } M); \\ \lim_{t \rightarrow \infty} \frac{1}{t} d_M(\gamma_1(t), \gamma_2(t)) & \text{(otherwise).} \end{cases} \quad (2.3.1)$$

Here d_M denotes the distance function on M . The relationship between these two distance functions is derived in [31, Proposition 2.2]. Namely, if rays γ_1 and γ_2 lie on the same end of M , then:

$$\tilde{d}_\infty([\gamma_1], [\gamma_2]) = \sqrt{2 - 2 \cos(d_\infty([\gamma_1], [\gamma_2]))}.$$

In particular, d_∞ and \tilde{d}_∞ induce the same topology on $M(\infty)$.

$(M(\infty), d_\infty)$ is an Alexandrov space with curvature bounded below by 1. The proof of this can be found in [15], and the definition and basic properties of Alexandrov spaces can be found in [4].

2.4 Basic properties of Riemannian submersions

In this section we review the definition and basic properties of Riemannian submersions, whose relevance to the study of nonnegative curvature is obvious from Perelman's Theorem. A good reference for this material is Chapter 9 of [2].

We begin with the definition. Suppose that M, B are Riemannian manifolds, and that $f : M \rightarrow B$ is a C^k submersion in the sense of differentiable topology; that is, f is onto and $df_x : T_x M \rightarrow T_{f(x)} B$ is a surjective linear map for every $x \in M$. This implies that for any $p \in B$, the fiber $F_p := f^{-1}(p)$ is a C^k submanifold of M . Define the vertical distribution, \mathcal{V} , of f to be the collection of tangent spaces of the fibers. That is, $\mathcal{V}_x := T_x(F_{f(x)})$. Define the horizontal distribution \mathcal{H} as the distribution orthogonal to \mathcal{V} . If for every $x \in M$, $df_x|_{\mathcal{H}_x} : \mathcal{H}_x \rightarrow T_{f(x)} B$ is a linear isometry, then f is called a Riemannian submersion.

If $\alpha(t)$ is a path in B , say from $p = \alpha(0)$ to $q = \alpha(1)$, and $x \in F_p$, then there exists a unique horizontal lift, $\bar{\alpha}_x(t)$ of α with $\bar{\alpha}_x(0) = x$. We define $h^\alpha : F_p \rightarrow F_q$ as the diffeomorphism between the fibers which sends each $x \in F_p$ to $\bar{\alpha}_x(1) \in F_q$. h^α is a C^{k-1} diffeomorphism, and we will sometimes refer to it as the “holonomy diffeomorphism” associated to α .

It is natural to define the “holonomy group”, Φ , of f by fixing a point $p \in B$ and defining Φ as the group of all diffeomorphisms of the fiber F_p which occur as h^α for some piecewise smooth loop α in B at p . It is easy to see that Φ is a group and that, up to group isomorphism, Φ does not depend on the choice of $p \in B$, at least when B is connected.

For the metric projection onto a soul, $\pi : M \rightarrow \Sigma$, the holonomy group of π is the same as the holonomy group of the normal bundle of the soul, and is therefore a subgroup of the orthogonal group $O(\nu_p(\Sigma))$. This is immediate from Perelman's Theorem. For general Riemannian submersions, however, the holonomy group need not be a finite dimensional Lie Group. In Chapter 4, we will study Riemannian submersions whose holonomy groups are compact finite dimensional Lie Groups. We will show by example that the holonomy group of the metric projection onto a soul need not be compact, and we will derive rigidity consequences when it is compact.

Associated to any Riemannian submersion are two fundamental tensors, called the A and T tensors. If E_1 and E_2 are vector fields on M , then we define:

$$T_{E_1} E_2 := \mathcal{H} \nabla_{\mathcal{V} E_1} \mathcal{V} E_2 + \mathcal{V} \nabla_{\mathcal{V} E_1} \mathcal{H} E_2.$$

$$A_{E_1} E_2 := \mathcal{H} \nabla_{\mathcal{H} E_1} \mathcal{V} E_2 + \mathcal{V} \nabla_{\mathcal{H} E_1} \mathcal{H} E_2.$$

It is easy to show that A and T are tensors. The T -tensor is best thought of as the second fundamental form of the fibers packaged together with its adjoint (the shape operator of the fibers). The best way to think of the A -tensor is provided by the following fact: if X and Y are horizontal vector fields on M then $A_X Y = \frac{1}{2} \mathcal{V}[X, Y]$. Thus, the A -tensor measures the failure of the horizontal distribution to be integrable.

Curvature information on M is related to curvature information on B by a collection of formulas known as O'Neill's formulas [2, Theorem 9.28]. The best known formula from this collection states that for $x \in M$, if $X, Y \in \mathcal{H}_x$ are orthonormal, then

$$K(X, Y) = K(df_x(X), df_x(Y)) - 3|A_X Y|^2. \quad (2.4.1)$$

Equation 2.4.2.1 is often referred to simply as “O'Neill's formula”. It implies the important fact that Riemannian submersions are curvature non-decreasing. It also implies that if M satisfies a lower curvature bound λ , and B satisfies an upper curvature bound μ , then the A -tensor of f is bounded in norm by $\sqrt{\frac{1}{3}(\mu - \lambda)}$. For the metric projection onto a soul, $\pi : M \rightarrow \Sigma$, M satisfies the lower curvature bound $\lambda = 0$, and Σ satisfies some upper curvature bound μ simply because Souls are compact. Therefore, the A -tensor of π is bounded in norm. Part 4 of Perelman's Theorem says that the T -tensor of π is also bounded in norm. For this reason, a large portion of our paper is devoted to the study of “bounded Riemannian submersions”; that is, Riemannian submersions both of whose fundamental tensors are bounded in norm.

Chapter 3

Conditions for nonnegative curvature on vector bundles

In the first two sections of this chapter, we use Perelman's Theorem to derive simple formulas which shed light on how the curvature tensor of M behaves at and near the soul of M . In the remaining sections, we use these formulas to translate the question of whether a vector bundle admits a metric of nonnegative curvature into the question of whether it admits a connection and a tensor which together satisfy a certain differential equation. This generalizes the work of Walschap and Strake on conditions for a vector bundle to admit a connection metric of nonnegative curvature [35].

3.1 Curvature at the soul

In this section, M will denote an open manifold of nonnegative curvature, and $\pi : M \rightarrow \Sigma$ will denote the metric projection onto its soul. It is obvious that π is smooth in a neighborhood of Σ . We use the curvature sign convention of [2]. For vector fields E_i on M , we denote by $(E_1, E_2, E_3, E_4) := \langle R(E_1, E_2)E_3, E_4 \rangle$, where R is the curvature tensor of M . We denote by $K(E_1, E_2) := (E_1, E_2, E_1, E_2)$. Notice that if E_1 and E_2 are orthonormal, then $K(E_1, E_2)$ is the sectional curvature of the 2-plane which they span.

For $p \in \Sigma$, $X, Y \in T_p\Sigma$, and $U, V \in \nu_p(\Sigma)$, we denote by $R^\nabla(X, Y)V$ the curvature tensor of $\nu(\Sigma)$. We can enlarge the domain of the tensor R^∇ somewhat by defining $R^\nabla(U, V)X$ to be the vector in $T_p\Sigma$ for which $\langle R^\nabla(U, V)X, Y \rangle = \langle R^\nabla(X, Y)U, V \rangle$ for any $Y \in T_p\Sigma$. Since Σ is totally geodesic, the tensor R^∇ is simply the restriction of R to a smaller domain; even so, we will often use R^∇ in order to point out that a formula depends only on the connection on $\nu(\Sigma)$.

When X is a vector field on Σ , we will use \tilde{X} to denote its basic lift to M .

We begin by reviewing a simple description of the A tensor of π . This description is due to Guijarro [14, Proposition 14], although the main idea first appeared in a paper by Strake and Walschap [34, Proposition 1.7]:

Lemma 3.1.1 (Guijarro). *Let $p \in \Sigma$, $X, Y \in T_p\Sigma$, and $W \in \nu_p(\Sigma)$. Let $\bar{p} := \exp^\perp(W)$. Then $A_{\tilde{X}(\bar{p})}\tilde{Y}(\bar{p}) = J(t)$, where $J(t)$ is the Jacobi field along $\gamma(t) := \exp^\perp(tW)$ with $J(0) = 0$ and $J'(0) = \frac{1}{2}R^\nabla(X, Y)W$.*

Proof. Write $T(\nu(\Sigma)) = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{V}}$ for the splitting of the tangent bundle of $\nu(\Sigma)$ into horizontal and vertical sub-bundles induced by the natural connection on $\nu(\Sigma)$. Remember that Perelman's Theorem implies that $(\exp^\perp)_* : \tilde{\mathcal{H}} \oplus \tilde{\mathcal{V}} \rightarrow \mathcal{H} \oplus \mathcal{V}$ preserves the splitting. Extend X, Y to vector fields on Σ in a neighborhood of p . Let \tilde{X}, \tilde{Y} denote their basic lifts to $T(\nu(\Sigma))$. Notice that

$(\exp^\perp)_* \tilde{X} = \bar{X}$ and $(\exp^\perp)_* \tilde{Y} = \bar{Y}$. Then:

$$A_{\bar{X}(\bar{p})} \bar{Y}(\bar{p}) = \frac{1}{2} [\bar{X}, \bar{Y}](\bar{p})^\vee = (\exp^\perp)_* \left(\frac{1}{2} [\tilde{X}, \tilde{Y}](W)^{\tilde{\vee}} \right) = (\exp^\perp)_* (\tilde{A}_{\tilde{X}(W)} \tilde{Y}(W)),$$

where \tilde{A} denotes the A -tensor associated with the splitting $\tilde{\mathcal{H}} \oplus \tilde{\mathcal{V}}$ of $T(\nu(\Sigma))$.

The rightmost term of the above equation can be simplified with the help of a description in [26, pages 67-70] of the A -tensor of a vector bundle with a connection. Namely:

$$\tilde{A}_{\tilde{X}(W)} \tilde{Y}(W) = \frac{1}{2} R^\nabla(X, Y)W. \quad (3.1.1)$$

In summary:

$$A_{\bar{X}(\bar{p})} \bar{Y}(\bar{p}) = \frac{1}{2} (\exp^\perp_*) (R^\nabla(X, Y)W),$$

so the proposition follows from the usual description of the derivative of the exponential map using Jacobi fields. \square

Remark 3.1.2. It is obvious from Lemma 3.1.1 that $A \equiv 0$ iff $R^\nabla \equiv 0$. But Walschap proved in [41, Theorem 1.3] that, for any Riemannian submersion the total space of which has nonnegative curvature, if $A \equiv 0$ then the submersion splits locally isometrically. This proves the local statement of Theorem 2.2.1 (The Splitting Theorem).

Cheeger and Gromoll proved in their original Soul Theorem paper that the curvatures of “mixed” 2-planes at the soul vanish [6, Theorem 3.1]. That is:

Lemma 3.1.3 (Cheeger-Gromoll). *Let $p \in \Sigma$, $X \in T_p \Sigma$ and $V \in \nu_p(\Sigma)$. Then $R(X, V)V = R(V, X)X = 0$. In particular $K(X, V) = 0$.*

Although today it is obvious from Perelman’s Theorem that $K(X, V) = 0$, Cheeger and Gromoll’s original argument was very difficult. The implication “ $K(X, V) = 0 \Rightarrow R(X, V)V = R(V, X)X = 0$ ” is a consequence of the following well-known lemma:

Lemma 3.1.4. *Let M be a manifold of nonnegative curvature, $p \in M$ and $X, V \in T_p M$ two vectors such that $K(X, V) = 0$. Then $R(X, V)V = R(V, X)X = 0$.*

Proof. Consider the map $F : T_p M \rightarrow T_p M$ defined as $F(Y) := -R(Y, V)V$. This map is symmetric, which means that $\langle F(Y), Z \rangle = \langle F(Z), Y \rangle$, and nonnegative, which means that $\langle F(Y), Y \rangle \geq 0$. From these two properties one can show that $\langle F(Y), Y \rangle = 0 \Leftrightarrow F(Y) = 0$. It follows that $R(X, V)V = 0$. Swapping X and V in the above argument proves that $R(V, X)X = 0$ as well. \square

Next we collect some facts about the A and T tensors and their derivatives at the soul. The following lemma says that A and T both vanish at points of Σ , as does the first derivative, DT , of T .

Lemma 3.1.5. *Let $p \in \Sigma$, $X, Y \in T_p \Sigma$, and $U, V, W \in \nu_p(\Sigma)$.*

1. $A_X Y = A_X U = 0$.
2. $T_U X = T_U V = 0$.
3. $(D_V A)_X Y = \frac{1}{2} R^\nabla(X, Y)V$ and $(D_V A)_X U = -\frac{1}{2} R^\nabla(V, U)X$.
4. $(D_W T)_U V = (D_W T)_U X = 0$

Proof. For part 1, $A_X Y = 0$ because the soul is totally geodesic. Since $\langle A_X U, Y \rangle = -\langle A_X Y, U \rangle = 0$, it follows that $A_X U = 0$ as well.

For part 2, $T_U X = 0$ because the “Perelman flat” through X and U is totally geodesic. Since $\langle T_U V, X \rangle = -\langle T_U X, V \rangle = 0$, it follows that $T_U V = 0$ as well.

The first statement of part 3 is an immediate consequence of Lemma 3.1.1. The second statement follows from the first as follows:

$$\langle (D_V A)_X U, Y \rangle = -\langle (D_V A)_X Y, U \rangle = -\langle \frac{1}{2} R^\nabla(X, Y) V, U \rangle = \langle -\frac{1}{2} R^\nabla(V, U) X, Y \rangle.$$

For part 4, first notice that $(D_W T)_W X = 0$ because the Perelman flat through X and W is totally geodesic. Next, since

$$0 = \langle (D_W T)_W X, U \rangle = -\langle (D_W T)_W U, X \rangle = -\langle (D_W T)_U W, X \rangle,$$

it follows that $(D_W T)_U W = 0$ as well. This is a special case of part 4. To get the general case from this special case, we apply O’Neill’s formula and Lemma 3.1.3 as follows:

$$0 = \langle X, W, W, U \rangle = \langle U, W, W, X \rangle = \langle (D_W T)_U W, X \rangle - \langle (D_U T)_W W, X \rangle = -\langle (D_U T)_W W, X \rangle.$$

It follows from this that $(D_U T)_W W = 0$. Since $(D_U T)_{W_1} W_2$ is symmetric in W_1 and W_2 , it follows that $D_U T = 0$. \square

Corollary 3.1.6. *Let $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $U \in \nu_p(\Sigma)$. Then $R(X, Y)U = 2R(X, U)Y$.*

Proof of Corollary 3.1.6. Since the soul is totally geodesic, both $R(X, Y)U$ and $R(X, U)Y$ are elements of $\nu_p(\Sigma)$. According to one of O’Neill’s formulas ([2, Theorem 9.28]), for any $V \in \nu_p(\Sigma)$,

$$\begin{aligned} \langle X, U, Y, V \rangle &= \langle (D_X T)_U V, Y \rangle - \langle T_U X, T_V Y \rangle + \langle (D_U A)_X Y, V \rangle + \langle A_X U, A_Y V \rangle \\ &= 0 + 0 + \frac{1}{2} \langle R^\nabla(X, Y) U, V \rangle + 0 = \frac{1}{2} \langle X, Y, U, V \rangle. \end{aligned}$$

\square

We wish to mention an alternative proof of this Corollary which was described to us by Peter Petersen. Unlike the proof above, it does not rely on Perelman’s Theorem.

Alternative proof of Corollary 3.1.6. From the vanishing of the mixed curvatures and the fact that the soul is totally geodesic, we see that:

$$0 = R(X + Y, U)(X + Y) = R(X, U)Y + R(Y, U)X + 0 + 0.$$

By the Bianchi identity:

$$R(X, Y)U = R(X, U)Y - R(Y, U)X.$$

From these two observations it follows that $R(X, Y)U = 2R(X, U)Y$. \square

We are now ready to study the sectional curvature of an arbitrary 2-plane at the soul. The following formula appears in [40, page 615]:

Proposition 3.1.7 (Walschap). *Let $p \in \Sigma$, $X, Y \in T_p \Sigma$, and $U, V \in \nu_p(\Sigma)$. Then:*

$$\boxed{K(X + U, Y + V) = K(X, Y) + K(U, V) + 3\langle X, Y, U, V \rangle}$$

Proof. Expanding the left hand side linearly gives:

$$\begin{aligned} K(X+U, Y+V) &= (X, Y, X, Y) + (X, V, X, V) + (U, Y, U, Y) + (U, V, U, V) + \\ &\quad 2\{(X, Y, X, V) + (X, Y, U, Y) + (X, Y, U, V) + \\ &\quad (X, V, U, Y) + (X, V, U, V) + (U, Y, U, V)\} \end{aligned}$$

Most of the 10 terms in this expression can be simplified. Specifically:

- $(X, V, X, V) = (U, Y, U, Y) = (X, V, U, V) = (U, Y, U, V) = 0$ by Lemma 3.1.3
- $(X, Y, X, V) = (X, Y, U, Y) = 0$ because the soul is totally geodesic.
- $(X, V, U, Y) = \frac{1}{2}(X, Y, U, V)$ by Corollary 3.1.6

□

The following corollary also appears in [40, page 615]:

Corollary 3.1.8 (Walschap). *Let $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $U, V \in \nu_p(\Sigma)$. Then:*

1. $3(X, Y, U, V) \leq K(X, Y) + K(U, V)$.

Equivalently,

2. $9(X, Y, U, V)^2 \leq 4K(X, Y) \cdot K(U, V)$.

Proof. Equation 1 just comes from Proposition 3.1.7, plus the fact that M has nonnegative curvature. Equation 2 follows from equation 1 as follows. Choose arbitrary vectors $X, Y \in T_p \Sigma$ and $U, V \in \nu_p(\Sigma)$. It is easy to see that if $K(X, Y) = 0$ then $(X, Y, U, V) = 0$; otherwise rescaling V by a suitably small constant would contradict equation 1, since (X, Y, U, V) depends linearly on $|V|$, while $K(U, V)$ depends quadratically on $|V|$. Similarly if $K(U, V) = 0$ then $(X, Y, U, V) = 0$. It remains to verify equation 2 when neither $K(X, Y)$ nor $K(U, V)$ is zero.

Choose λ_1, λ_2 so that $K(X, \lambda_1 Y) = K(U, \lambda_2 V)$. Then $3(X, \lambda_1 Y, U, \lambda_2 V) \leq K(X, \lambda_1 Y) + K(U, \lambda_2 V) = 2K(X, \lambda_1 Y)$, so $9(X, \lambda_1 Y, U, \lambda_2 V)^2 \leq 4K(X, \lambda_1 Y)^2 = 4K(X, \lambda_1 Y)K(U, \lambda_2 V)$. In other words, equation 2 is true for the vectors $X, \lambda_1 Y, U, \lambda_2 V$. But since equation 2 is scale-invariant, it must also be true of the vectors X, Y, U, V .

In fact equations 1 and 2 are equivalent. To see that equation 2 implies equation 1, just notice that for any real numbers A, B, C , if $A, B \geq 0$ and $C^2 \leq 4AB$ then $C \leq A + B$. To see this, set $\lambda := A/B$. Since $C^2 \leq 4AB = 4\lambda B^2$,

$$C \leq 2\sqrt{\lambda}B \leq (1 + \lambda)B = A + B.$$

□

The following splitting theorem was observed independently by Marenich [20, Theorem 2] and Walschap [40, Proposition 2.6]:

Corollary 3.1.9. *If for every $p \in \Sigma$ and for every $U, V \in \nu_p(\Sigma)$, $K(U, V) = 0$, then M splits locally isometrically over its soul.*

Proof. If the vertical curvatures vanish, then the previous corollary implies that the curvature tensor R^∇ vanishes, and the splitting now follows from Theorem 2.2.1. □

3.2 Curvature near the soul

In the preceding section we derived a formula for the curvature of an arbitrary 2-plane at a point $p \in \Sigma$. In this section we study the derivatives of the function which records the curvatures of a family of 2-planes initially based at the soul but then drifting away from the soul. The family will begin with a mixed 2-plane at the soul (which has zero curvature by Lemma 3.1.3), and will then drift so that the base point moves away from the soul while simultaneously the 2-plane twists away from being a mixed 2-plane.

More precisely, the set up for this section is as follows. Let $p \in \Sigma$. Let $X, Y \in T_p \Sigma$ and let $W, V, U \in \nu_p(\Sigma)$. Let $\gamma(t) := \exp^\perp(tW)$, and let X_t, Y_t, U_t, V_t denote the parallel transports of X, Y, U and V along $\gamma(t)$. By Perelman's Theorem, X_t and Y_t are horizontal for all $t \in [0, \infty)$. In other words, parallel translation along the radial geodesic γ preserves the horizontal space. It therefore must also preserve the vertical space, and hence U_t and V_t are vertical for all $t \in [0, \infty)$. Define:

$$K(t) := K_{W,X,V,Y,U}(t) := K(X_t + tU_t, tY_t + V_t),$$

which is the unnormalized sectional curvature of the 2-plane based at $\gamma(t)$ spanned by $X_t + tU_t$ and $tY_t + V_t$. The special case of this construction when $U = Y = 0$ was studied by Marenich in [20]. Notice that $K(0) = K(X, V) = 0$ by Lemma 3.1.3. The goal of this section is to derive formulas for $K'(0)$ and $K''(0)$. Towards this end, we write:

$$\begin{aligned} K(t) &= (X_t, V_t, X_t, V_t) + t \cdot \{2(X_t, Y_t, X_t, V_t) + 2(X_t, V_t, U_t, V_t)\} \\ &\quad + t^2 \cdot \{(X_t, Y_t, X_t, Y_t) + (U_t, V_t, U_t, V_t) + 2(X_t, Y_t, U_t, V_t) + 2(X_t, V_t, U_t, Y_t)\} \\ &\quad + t^3 \cdot \{2(U_t, Y_t, U_t, V_t) + 2(X_t, Y_t, U_t, Y_t)\} + t^4 \cdot (U_t, Y_t, U_t, Y_t) \end{aligned} \quad (3.2.1)$$

Our calculations will require the following Lemma:

Lemma 3.2.1. *Let M be a Riemannian manifold and let A denote a tensor of order k on M (that is, for each $p \in M$, $A_p : (T_p M)^k \rightarrow \mathbb{R}$). For $p \in M$ and $X, Y \in T_p M$, if $A_p = 0$ then $(D_X D_Y A)_p = (D_Y D_X A)_p$.*

Proof. The proof is straightforward and is left to the reader □

Proposition 3.2.2. $K'(0) = 0$

Proof. Since M has nonnegative curvature, $K'(0) \geq 0$. But if it were the case that $K'(0) > 0$, then replacing W with $-W$ would yield $K'(0) < 0$. Hence $K'(0) = 0$.

This provides an a priori reason to expect that $K'(0) = 0$. In order that our proof generalizes properly in the next section, we also compute $K'(0)$ directly. From Equation 3.2.3.1, we see that

$$K'(0) = 2(X, Y, X, V) + 2(X, V, U, V) + \left. \frac{d}{dt} \right|_{t=0} (X_t, V_t, X_t, V_t).$$

But $(X, Y, X, V) = 0$ because the soul is totally geodesic, and $(X, V, U, V) = 0$ by Lemma 3.1.3. We use one of O'Neill's formulas to study the third term:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (X_t, V_t, X_t, V_t) &= \left. \frac{d}{dt} \right|_{t=0} \{ \langle (D_{X_t} T)_{V_t} V_t, X_t \rangle - \langle T_{V_t} X_t, T_{V_t} X_t \rangle + \langle A_{X_t} V_t, A_{X_t} V_t \rangle \} \\ &= \left. \frac{d}{dt} \right|_{t=0} \langle (D_{X_t} T)_{V_t} V_t, X_t \rangle - 0 + 0 \\ &= \left\langle \frac{D}{dt} ((D_{X_t} T)_{V_t} V_t), X \right\rangle = \langle (D_W D_X T)_V V, X \rangle = \langle (D_X D_W T)_V V, X \rangle = 0. \end{aligned}$$

In the second equality above, the second and third terms vanish because, by Leibnitz' rule, the derivative at time $t=0$ of the inner product of two vector fields both of which vanish at $t=0$ must be zero. □

Proposition 3.2.3.

$$\begin{aligned} K''(0) &= 2K(X+U, Y+V) + 2\langle (D_X R^\nabla)(X, Y)W, V \rangle + \frac{1}{2}|R^\nabla(W, V)X|^2 \\ &\quad + \langle 4(D_W D_U T)_V V - 4(D_W D_V T)_U V + (D_X D_W^2 T)_V V, X \rangle \end{aligned}$$

Proof. From Equation 3.2.3.1, we see that:

$$\begin{aligned} K''(0) &= 2\{(X, Y, X, Y) + (U, V, U, V) + 2(X, Y, U, V) + 2(X, V, U, Y)\} \\ &\quad + 2\frac{d}{dt}\Big|_{t=0} \{2(X_t, Y_t, X_t, V_t) + 2(X_t, V_t, U_t, V_t)\} \\ &\quad + \frac{d^2}{dt^2}\Big|_{t=0} (X_t, V_t, X_t, V_t). \end{aligned}$$

- The top line of this expression is familiar from the proof of Proposition 3.2.2:

$$\begin{aligned} &(X, Y, X, Y) + (U, V, U, V) + 2(X, Y, U, V) + 2(X, V, U, Y) \\ &= K(X, Y) + K(U, V) + 3(X, Y, U, V) \\ &= K(X+U, Y+V) \end{aligned}$$

- Next, from one of O'Neill's formulas:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} (X_t, Y_t, X_t, V_t) &= \frac{d}{dt}\Big|_{t=0} \{\langle (D_{X_t} A)_{X_t} Y_t, V_t \rangle + 2\langle A_{X_t} Y_t, T_{V_t} X_t \rangle\} \\ &= \frac{d}{dt}\Big|_{t=0} \langle (D_{X_t} A)_{X_t} Y_t, V_t \rangle = \langle \frac{D}{dt}((D_{X_t} A)_{X_t} Y_t), V \rangle \\ &= \langle (D_W D_X A)_X Y, V \rangle = \langle (D_X D_W A)_X Y, V \rangle \\ &= \frac{1}{2}\langle (D_X R^\nabla)(X, Y)W, V \rangle \end{aligned}$$

- We apply another of O'Neill's formulas to simplify the next term:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} (X_t, V_t, U_t, V_t) &= -\frac{d}{dt}\Big|_{t=0} (U_t, V_t, V_t, X_t) \\ &= -\frac{d}{dt}\Big|_{t=0} \{\langle (D_{V_t} T)_{U_t} V_t, X_t \rangle - \langle (D_{U_t} T)_{V_t} V_t, X_t \rangle\} \\ &= -\langle (D_W D_V T)_U V, X \rangle + \langle (D_W D_U T)_V V, X \rangle \end{aligned}$$

- Finally,

$$\begin{aligned} &\frac{d^2}{dt^2}\Big|_{t=0} (X_t, V_t, X_t, V_t) \\ &= \frac{d^2}{dt^2}\Big|_{t=0} \{\langle (D_{X_t} T)_{V_t} V_t, X_t \rangle - \langle T_{V_t} X_t, T_{V_t} X_t \rangle + \langle A_{X_t} V_t, A_{X_t} V_t \rangle\} \\ &= \langle (D_W^2 D_X T)_V V, X \rangle - 2\langle (D_W T)_V X, (D_W T)_V X \rangle + 2\langle (D_W A)_X V, (D_W A)_X V \rangle \\ &= \langle (D_X D_W^2 T)_V V, X \rangle - 0 + \frac{1}{2}|R^\nabla(W, V)X|^2. \end{aligned}$$

□

Corollary 3.2.4. *If π has totally geodesic fibers (or more generally if the T -tensor of π vanishes to second order at Σ), then for all $X, Y \in T_p \Sigma$ and all $V, W \in \nu_p(\Sigma)$:*

$$1. \quad 2\langle (D_X R^\nabla)(X, Y)W, V \rangle \leq \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, Y).$$

Equivalently,

$$2. \quad \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq |R^\nabla(W, V)X|^2 \cdot K(X, Y).$$

Proof. For equation 1, set $U = 0$ in Proposition 3.2.3, and observe that $K''(0) \geq 0$ simply because M has nonnegative curvature. Equations 1 and 2 are equivalent by the argument of Corollary 3.1.8. \square

Corollary 3.2.4 should be compared to [35, Corollary 3.2].

3.3 Connection metrics on vector bundles

By a connection metric on a vector bundle $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$, we mean a metric which arises by choosing the following data:

1. A metric g_Σ on Σ .
2. A Euclidean structure on the bundle (which means a smoothly varying inner product on the fibers).
3. A connection ∇ on the bundle which is compatible with the Euclidean structure.
4. A rotationally symmetric metric g_0 on \mathbb{R}^k .

Then there is a unique metric g_E on E (called a connection metric) for which $\pi : (E, g_E) \rightarrow (\Sigma, g_\Sigma)$ is a Riemannian submersion whose horizontal distribution is determined by ∇ and such that each fiber is totally geodesic and isometric to (\mathbb{R}^k, g_0) .

An arbitrary rotationally symmetric metric g_0 on \mathbb{R}^k can be expressed in polar coordinates as $ds^2 = dr^2 + G^2(r)d\Theta^2$, where $G(r)$ is a function for which $G(0) = 0$, $G'(0) = 1$, and all even derivatives of G vanish at 0. The curvature of any 2-plane at the origin of (\mathbb{R}^k, g_0) equals $-G'''(0)$.

Given such a connection metric, we will denote by R^∇ the curvature tensor associated to the connection ∇ , and by R the curvature tensor of (E, g_E) . By definition, the T -tensor of $\pi : (E, g_E) \rightarrow (\Sigma, g_\Sigma)$ vanishes. The A -tensor of this Riemannian submersion can be described as follows. For all $p \in \Sigma$, $X, Y \in T_p \Sigma$, and $W \in E_p = \pi^{-1}(p)$, if \bar{X} and \bar{Y} denote the horizontal lifts of X and Y to $T_W E$, then, as mentioned in Equation 3.1.3.1:

$$A(\bar{X}, \bar{Y}) = \frac{1}{2}R^\nabla(X, Y)W. \quad (3.3.1)$$

Also as before, for $X, Y \in T_p \Sigma$ and $U, V, W \in E_p := \pi^{-1}(p)$, denote $K(t) = K_{W, X, V, Y, U}(t) = K(X_t + tU_t, tY_t + V_t)$, where X_t, Y_t, U_t, V_t are the parallel translation of X, Y, U, V along the geodesic in the direction of W .

It is still unknown which vector bundles over spheres (or more general souls) admit connection metrics of nonnegative curvature. In [35], Strake and Walschap studied this question. They derived conditions on $\{g_B, \nabla, g_0\}$ under which such a connection metric will have nonnegative curvature. Their main result is as follows:

Theorem 3.3.1 (Strake-Walschap). *Suppose (Σ, g_Σ) is a compact Riemannian manifold with strictly positive sectional curvature. Suppose that $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$ is a vector bundle over Σ with a Euclidean structure and connection ∇ . If there exists $\epsilon > 0$ such that for each $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p := \pi^{-1}(p)$, the following conditions hold:*

$$1. \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (\tfrac{1}{2} - \epsilon) \cdot |R^\nabla(W, V)X|^2 \cdot K(X, Y).$$

2. If $R^\nabla(W, V)X = 0$ then $X = 0$ or W is parallel to V .

then a rotationally symmetric metric g_0 on \mathbb{R}^k can be chosen such that the corresponding connection metric g_E on E has nonnegative sectional curvature.

In this section we will use the formulas and observations of the preceding two sections to obtain results similar to Strake and Walschap's above theorem.

Perelman's Theorem demonstrates that an arbitrary open manifold of nonnegative curvature is much more similar than previously expected to the total space of a vector bundle with a connection metric. For example, the first statement of Perelman's Theorem (the existence of Perelman flats) is true for any connection metric on any vector bundle:

Lemma 3.3.2. *Let $\pi : E \rightarrow \Sigma$ denote a vector bundle over Σ with a connection metric.*

1. *The zero-section, $\Sigma \subset E$, is totally geodesic.*
2. *Let $p \in \Sigma$, $X \in T_p \Sigma$, and $V \in E_p$. Then the surface $(s, t) \mapsto \exp(s \cdot V(t))$ ($s, t \in \mathbb{R}, s \geq 0$), where $V(t)$ denotes the parallel transport of V along the geodesic with initial tangent vector X , is a flat and totally geodesic half-plane.*
3. *For sufficiently small r , the closed ball $\bar{B}_\Sigma(r)$ is convex.*

Proof. The first two claims are elementary. The third follows from the second via the argument in the proof of Proposition 2.1.5. \square

It should be unsurprising that most of the formulas from the previous two sections are also true of vector bundles with connection metrics. Specifically:

Proposition 3.3.3. *Let $R^k \rightarrow E \xrightarrow{\pi} \Sigma$ denote a vector bundle with a connection metric. Let $p \in \Sigma$, $X, Y \in T_p \Sigma$, and $U, V, W \in E_p := \pi^{-1}(p)$. Then:*

1. $A_X Y = A_X U = 0$.
2. $(D_V A)_X Y = \tfrac{1}{2} R^\nabla(X, Y)V$ and $(D_V A)_X U = -\tfrac{1}{2} R^\nabla(V, U)X$.
3. $R(X, V)V = R(V, X)X = 0$ and $R(X, Y)U = 2R(X, U)Y$.
4. $K(X + U, Y + V) = K(X, Y) + K(U, V) + 3\langle X, Y, U, V \rangle$.
5. $K'_{W, X, V, Y, U}(0) = 0$.
6. $K''_{W, X, V, Y, U}(0) = 2K(X + U, Y + V) + 2\langle (D_X R^\nabla)(X, Y)W, V \rangle + \tfrac{1}{2}|R^\nabla(W, V)X|^2$.

Proof. Item 1 is obvious. Item 2 is clear from Equation 3.3.3.1. For the first part of item 3, use O'Neill's formulas plus the vanishing of the T -tensor to show that $(X, V, V, Y) = (X, V, V, U) = (V, X, X, Y) = (V, X, X, U) = 0$. Then the second part of item 3 follows from the first by the argument of Corollary 3.1.6. Items 4, 5, and 6 are proven exactly like their analogs in the previous two sections. \square

We will ask a slightly easier question than did Strake and Walschap. Instead of seeking a connection metric on a bundle which has nonnegative curvature, we will only seek a connection metric which, up to second order calculations at the soul, looks like it has nonnegative curvature. This notion is captured by the following definition:

Definition 3.3.4. A metric g_E on the total space E of a vector bundle $\mathbb{R}^k \rightarrow E \rightarrow \Sigma$ is said to be "nonnegatively curved to second order" if:

1. The zero section $\Sigma \subset E$ is totally geodesic.
2. For every $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $U, V \in E_p$, $K(X + U, Y + V) \geq 0$ and $= 0$ iff X is parallel to Y and U is parallel to V .
3. If $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $U, V, W \in E_p$, then $K'_{W,X,V,Y,U}(t) = 0$ and $K''_{W,X,V,Y,U}(0) \geq 0$. Further, when $W, X, V \neq 0$, $K''_{W,X,V,Y,U}(0) = 0$ iff V and U are both parallel to W , and Y is parallel to X , in which case $K_{W,X,V,Y,U}(t) = 0$ for all t .

Item 2 of this definition says that all 2-planes at Σ have nonnegative sectional curvature, and only the mixed 2-planes are flat. Therefore, if our goal is to insure that a neighborhood of Σ in E is nonnegatively curved, then the only trouble spot is a neighborhood (in the Grassmannian of 2-planes on M , which we denote by G) of the set, Z , of mixed 2-planes at Σ . Item 3 says that along any “straight line” $l(t)$ in G beginning in Z , the curvature function $K(t) = K(l(t))$ satisfies $K'(0) = 0$ and $K''(0) \geq 0$ and $= 0$ iff all of $l(t)$ lies on a single “Perelman flat”. Thus, a metric which is nonnegatively curved to second order is a very good candidate for having nonnegative curvature in a neighborhood of its zero-section. We do not know whether a connection metric which is nonnegatively curved to second order is necessarily nonnegatively curved in a neighborhood of its zero-section. This issue is of interest because of the following proposition:

Proposition 3.3.5. *Suppose that $E \xrightarrow{\pi} \Sigma$ is a vector bundle with a connection metric g_E on E which is nonnegatively curved in a neighborhood of the zero-section $\Sigma \subset E$. Then there exists a complete metric g'_E on E which is everywhere nonnegatively curved.*

Proof. According to part 3 of Lemma 3.3.2, $\bar{B}_\Sigma(r)$ is convex for sufficiently small $r > 0$. The statement then follows from [12, Theorem A] (or from [22, Theorem 1.2], on which the previous reference is based). In fact, it is easy to see from Guijarro’s construction that the new metric g'_E will also be a connection metric. \square

Thus, a connection metric which is nonnegatively curved to second order can very likely be deformed to a connection metric of nonnegative curvature. This motivates our interest in Definition 3.3.4. We prove the following:

Theorem 3.3.6. *Suppose (Σ, g_Σ) is a compact Riemannian manifold with strictly positive curvature, and $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$ is a vector bundle over Σ . Then there exists a connection metric g_E on E with soul (Σ, g_Σ) which has nonnegative curvature to second order iff the bundle admits a Euclidean structure and a connection ∇ such that the following condition holds for each $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p$:*

$$(*) \quad \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq |R^\nabla(W, V)X|^2 \cdot K(X, Y) \text{ with equality iff } Y \text{ is parallel to } X \text{ or } W \text{ is parallel to } V.$$

Lemma 3.3.7. *Condition $(*)$ of Theorem 3.3.6 is equivalent to:*

$$(**) \quad 2\langle (D_X R^\nabla)(X, Y)W, V \rangle \leq \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, Y). \text{ Further, when } X, W, V \neq 0, \text{ there is equality iff } Y \text{ is parallel to } X \text{ and } W \text{ is parallel to } V.$$

Proof. The inequalities in conditions $(*)$ and $(**)$ are equivalent by Corollary 3.2.4, so it remains only to discuss the equality cases. First we prove that $(**)$ implies $(*)$. Setting $Y = 0$ in $(**)$ implies the following:

$$\text{If } R^\nabla(W, V)X = 0 \text{ then } X = 0 \text{ or } W \text{ is parallel to } V. \quad (3.3.2)$$

Suppose that $\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 = |R^\nabla(W, V)X|^2 \cdot K(X, Y)$. Suppose also that Y is not parallel to X , and W is not parallel to V . Since Σ has strictly positive curvature, $K(X, Y) \neq 0$.

By 3.3.3.2, $|R^\nabla(W, V)X|^2 \neq 0$. Choose λ_1, λ_2 such that $|R^\nabla(W, \lambda_2 V)X|^2 = 4 \cdot K(X, \lambda_1 Y)$. Since condition (*) is invariant under rescaling of the vectors,

$$\langle (D_X R^\nabla)(X, \lambda_1 Y)W, \lambda_2 V \rangle^2 = |R^\nabla(W, \lambda_2 V)X|^2 \cdot K(X, \lambda_1 Y) = 4K(X, \lambda_1 Y)^2,$$

and hence,

$$\langle (D_X R^\nabla)(X, \lambda_1 Y)W, \lambda_2 V \rangle = 2K(X, \lambda_1 Y) = \frac{1}{4}|R^\nabla(W, \lambda_2 V)X|^2 + K(X, \lambda_1 Y),$$

which contradicts (**).

Next we prove that (*) implies (**). First notice that (*) implies 3.3.3.2. Assume that $2\langle (D_X R^\nabla)(X, Y)W, V \rangle = \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, Y)$, and that $X, V, W \neq 0$. Since Σ has strictly positive curvature, if W is parallel to V then X must be parallel to Y . By 3.3.3.2, if X is parallel to Y , then W must be parallel to V . So it remains to contradict the hypothesis that X is not parallel to Y AND W is not parallel to V . Assume that this is the case. Define:

$$f(t) := -2\langle (D_X R^\nabla)(X, tY)W, V \rangle + \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, tY).$$

$f(t)$ is a nonnegative valued quadratic function with $f(1) = 0$, and hence its discriminant equals 0. This means that $\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 = |R^\nabla(W, V)X|^2 \cdot K(X, Y)$, which contradicts (*). \square

Lemma 3.3.8. *Conditions (*) and (**) are also equivalent to the following equivalent conditions:*

- (* ϵ) For some $\epsilon > 0$, $\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (1 - \epsilon)|R^\nabla(W, V)X|^2 \cdot K(X, Y)$ with equality iff Y is parallel to X or W is parallel to V .
- (** ϵ) For some $\epsilon > 0$, $2\langle (D_X R^\nabla)(X, Y)W, V \rangle \leq \frac{1}{2}|R^\nabla(W, V)X|^2 + 2(1 - \epsilon)K(X, Y)$. Further, when $X, W, V \neq 0$, there is equality iff Y is parallel to X and W is parallel to V .

Proof. Clearly (* ϵ) \Rightarrow (*) and (** ϵ) \Rightarrow (**). Also, (* ϵ) \Leftrightarrow (** ϵ) by the argument of the previous lemma. So it remains to prove that (*) \Rightarrow (* ϵ). Assuming (*), a compactness argument provides an $\epsilon > 0$ for which

$$\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (1 - \epsilon)|R^\nabla(W, V)X|^2 \cdot K(X, Y)$$

whenever $|X| = |Y| = |W| = |V| = 1$, $X \perp Y$ and $W \perp V$. But this equation is not sensitive to rescallings of the vectors, nor is it sensitive to replacing Y (respectively V) with its component orthogonal to X (respectively W). Hence, this equation holds for all X, Y, W, V with no restrictions. \square

Proof of Theorem 3.3.6. If the bundle admits a connection metric which is nonnegatively curved to second order, then condition (**) is satisfied by part 6 of Proposition 3.3.3. Conversely, suppose that condition (** ϵ) is satisfied. We can choose a radially symmetric metric g_0 on \mathbb{R}^k such that the curvature of every 2-plane at the origin of (\mathbb{R}^k, g_0) is arbitrarily large; in particular, we can make these curvatures larger than:

$$\sup \left\{ \frac{9(X, Y, U, V)^2}{4\epsilon K(X, Y)} \mid X, Y \in T_p \Sigma \text{ and } U, V \in E_p \text{ orthonormal} \right\}$$

Then $9(X, Y, U, V)^2 \leq 4\epsilon K(X, Y) \cdot K(U, V)$ for all orthonormal vectors X, Y, U, V , and hence for all (not necessarily orthonormal) vectors as well. This is equivalent to the following inequality:

$$3(X, Y, U, V) \leq \epsilon K(X, Y) + K(U, V) \quad (3.3.3)$$

(see the discussion of equivalence in Corollary 3.1.8). So by part 4 of Proposition 3.3.3, all 2-planes at Σ are nonnegatively curved, and it is easy to see that only the mixed 2-planes are flat.

Next, combining (** ϵ) with Equation 3.3.3.3 gives:

$$\begin{aligned} 2\langle (D_X R^\nabla)(X, Y)W, V \rangle &\leq \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, Y) - 2\epsilon K(X, Y) \\ &\leq \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, Y) - 2\{3\langle X, Y, U, V \rangle - K(U, V)\} \\ &= \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X - U, Y + V). \end{aligned}$$

It follows from part 6 of Proposition 3.3.3 that $K''_{W, X, V, Y, U}(0) \geq 0$. Assuming that $W, X, V \neq 0$, the second inequality above becomes an equality iff Y is parallel to X and U is parallel to V . In this case, the first inequality above becomes an equality iff V and W are parallel. \square

3.4 The warping tensor

As before, let M be an open manifold of nonnegative curvature and let $\pi : M \rightarrow \Sigma$ denote the metric projection onto its soul. We define the “warping function” and the “warping tensor” of M , both of which reflect how the metric on M near Σ differs from the connection metric with flat fibers on $\nu(\Sigma)$. The second derivatives of the T -tensor of π (which appear in Proposition 3.2.3) can be expressed nicely in terms of the warping tensor.

For $p \in \Sigma$ and $W, U, V \in \nu_p(\Sigma)$, define:

$$F(W, U, V) := \langle (d \exp^\perp)_W U, (d \exp^\perp)_W V \rangle.$$

We will call F the *warping function* of M . Since the metric on M is smooth, F is smooth in the domain where $|W| < \text{the cut-distance of } \Sigma$. The following properties of F are obvious from the definition:

Lemma 3.4.1. *For any $W, U, V \in \nu_p(\Sigma)$,*

1. $F(0, U, V) = \langle U, V \rangle$
2. *If $|W| < \text{the cut-distance of } \Sigma$, then $F(W, \cdot, \cdot)$ is a symmetric positive-definite bilinear form.*
3. $F(W, W, U) = \langle W, U \rangle$.
4. $\frac{d}{dt}|_{t=0} F(tW, U, V) = 0$
5. $\frac{d^2}{dsdt}|_{s=t=0} F(tW_1 + sW_2, U, V) = \frac{d^2}{dsdt}|_{s=t=0} F(tU + sV, W_1, W_2)$.

The last two properties follow from the smoothness of the metric (or equivalently from the smoothness of F).

Next, for every quadruple (W_1, W_2, U, V) of vectors in $\nu_p(\Sigma)$, define:

$$\Upsilon(W_1, W_2, U, V) := \frac{d^2}{dsdt} \Big|_{s=t=0} F(tW_1 + sW_2, U, V).$$

We will call Υ the *warping tensor* of M . It records second derivative information about how the metric on M differs near the soul from the connection metric with flat fibers on $\nu(\Sigma)$. For fixed U, V , think of Υ as the second derivative of the function from $\nu_p(\Sigma)$ to \mathbb{R} which sends $W \mapsto F(W, U, V)$. The following properties of Υ are clear from definition:

Lemma 3.4.2.

1. Υ is multi-linear, symmetric in (W_1, W_2) , and symmetric in (U, V) .
2. $\Upsilon(W, W, W, U) = 0$.
3. $\Upsilon(W_1, W_2, U_1, U_2) = \Upsilon(U_1, U_2, W_1, W_2)$

Our next goal is to express the second derivatives of the T -tensor (which appear in Proposition 3.2.3) in terms of Υ . In order to state the result, we need to establish some notation. We describe a method for extending a vector $U \in \nu_p(\Sigma)$ to a vertical vector field \bar{U} on a neighborhood of p in M . First, for $q \in \Sigma$ near p , define \hat{U}_q as the parallel transport of U to q along the minimal geodesic from p to q in Σ . This defines an extension \hat{U} of U to a neighborhood of p in Σ . Then for each $q \in \Sigma$ near p , extend the vector \hat{U}_q to a vertical vector field \bar{U} on the fiber F_q by defining $\bar{U}_{\exp^\perp(W)} := (d\exp^\perp)_W \hat{U}_q$ for all vectors $W \in \nu_q(\Sigma)$ with small norm.

Additionally, when $X \in T_p\Sigma$, we will denote by \bar{X} an extension of X to a basic vector field on M in a neighborhood of p .

For fixed vectors $W, U, V \in \nu_p(\Sigma)$, we can think of $F(W, U, V)$ as a real-valued function on a neighborhood of p in Σ , by parallel transporting W, U, V along radial geodesics from p . In other words, the function is $q \mapsto F(\hat{W}_q, \hat{U}_q, \hat{V}_q)$. With this interpretation, it makes sense to write $\text{grad}F(W, U, V) \in T_p\Sigma$ for the gradient of this function and $XF(W, U, V)$ for the derivative of this function in the direction $X \in T_p\Sigma$. Similarly, $\Upsilon(W_1, W_2, U, V)$ can be thought of as a real-valued function on a neighborhood of p in Σ , and we can write $\text{grad}\Upsilon(W_1, W_2, U, V)$ or $X\Upsilon(W_1, W_2, U, V) = D_X\Upsilon(W_1, W_2, U, V)$. By $\text{hess}_{\Upsilon(W_1, W_2, U, V)}(X)$ we mean the hessian of this function in the direction $X \in T_p\Sigma$.

Lemma 3.4.3. *Let $W, W_1, W_2, U, V \in \nu_p(\Sigma)$. Assume $|W| < \text{the cut-distance of } \Sigma$. Let $\bar{p} := \exp^\perp(W)$.*

1. $\langle T_{\bar{V}}\bar{X}, \bar{U} \rangle_{\bar{p}} = \frac{1}{2}XF(W, U, V)$ for any $X \in T_p\Sigma$.
2. $\langle T_{\bar{V}}\bar{U} \rangle_{\bar{p}} = -\frac{1}{2}\overline{\text{grad}F(W, U, V)}$.
3. $\langle D_{W_1}D_{W_2}T \rangle_U V = -\frac{1}{2}\text{grad}\Upsilon(W_1, W_2, U, V)$.

Proof. First notice that \bar{X} and \bar{U} commute simply because their preimages under $d\exp^\perp$ in $T(\nu(\Sigma))$ commute. So, using the standard coordinate-free expression for the connection,

$$\begin{aligned} 2\langle T_{\bar{V}}\bar{X}, \bar{U} \rangle_{\bar{p}} &= 2\langle \nabla_{\bar{V}}\bar{X}, \bar{U} \rangle_{\bar{p}} \\ &= \bar{X}\langle \bar{V}, \bar{U} \rangle_{\bar{p}} + \bar{V}\langle \bar{U}, \bar{X} \rangle_{\bar{p}} - \bar{U}\langle \bar{X}, \bar{V} \rangle_{\bar{p}} \\ &\quad - \langle [\bar{X}, \bar{U}], \bar{V} \rangle_{\bar{p}} - \langle [\bar{U}, \bar{V}], \bar{X} \rangle_{\bar{p}} - \langle [\bar{X}, \bar{V}], \bar{U} \rangle_{\bar{p}} \\ &= \bar{X}\langle \bar{V}, \bar{U} \rangle_{\bar{p}} = XF(W, U, V). \end{aligned}$$

This proves part 1. Part 2 is proven as follows:

$$\begin{aligned} \langle T_{\bar{V}}\bar{U}, \bar{X} \rangle_{\bar{p}} &= -\langle T_{\bar{V}}\bar{X}, \bar{U} \rangle_{\bar{p}} = -\frac{1}{2}XF(W, U, V) \\ &= \langle -\frac{1}{2}\text{grad}F(W, U, V), \bar{X} \rangle_{\bar{p}} = \langle -\frac{1}{2}\overline{\text{grad}F(W, U, V)}, \bar{X} \rangle_{\bar{p}}. \end{aligned}$$

For part 3, it will suffice to prove that $(D_W^2T)_U V = -\frac{1}{2}\text{grad}\Upsilon(W, W, U, V)$, which is done as follows. Let $\gamma(t) = \exp^\perp(tW)$. Then:

$$\begin{aligned}
D_W(D_W T_U V) &= \left. \frac{D}{dt} \right|_{t=0} (D_{\bar{W}} T_{\bar{U}} \bar{V})_{\gamma(t)} - 0 - 0 \\
&= \left. \frac{D}{dt} \right|_{t=0} \left(\left. \frac{D}{dr} \right|_{r=0} (T_{\bar{U}} \bar{V})_{\gamma(t+r)} - (T_{(\nabla_{\bar{W}} \bar{U})} \bar{V})_{\gamma(t)} - (T_{\bar{U}} (\nabla_{\bar{W}} \bar{V}))_{\gamma(t)} \right) \\
&= \left. \frac{D^2}{dt^2} \right|_{t=0} (T_{\bar{U}} \bar{V})_{\gamma(t)} - 0 - 0 \\
&= -\left. \frac{1}{2} \frac{D^2}{dt^2} \right|_{t=0} (\overline{\text{grad} F(tW, U, V)})_{\gamma(t)} = -\left. \frac{1}{2} \frac{D^2}{dt^2} \right|_{t=0} \text{grad} F(tW, U, V) \\
&= -\frac{1}{2} \text{grad} \left(\left. \frac{d^2}{dt^2} \right|_{t=0} F(tW, U, V) \right) = -\frac{1}{2} \text{grad} \Upsilon(W, W, U, V).
\end{aligned}$$

To justify the third equality above, notice that:

$$\begin{aligned}
\left. \frac{D}{dt} \right|_{t=0} (T_{(\nabla_{\bar{W}} \bar{U})} \bar{V})_{\gamma(t)} &= (D_W T)_{(\nabla_W \bar{U})} V + T_{(\nabla_W \nabla_{\bar{W}} \bar{U})} V + T_{(\nabla_W \bar{U})} (\nabla_W \bar{V}) \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

□

Next, we describe how to calculate the curvature of a 2-plane at the soul in terms of the warping tensor.

Lemma 3.4.4. *Let $W, V \in \nu_p(\Sigma)$. Then $K(W, V) = -\frac{3}{2} \Upsilon(W, W, V, V)$.*

Proof. It will suffice to prove this when W, V are orthonormal. Let S denote the surface in M obtained as the exponential image of the plane spanned by W and V . $K(W, V)$ is the Gauss curvature of S at p . Write the metric on S in polar coordinates:

$$ds^2 = dr^2 + f^2(r, \theta) d\theta^2,$$

where $\theta = 0$ corresponds to the direction of W . Let $\gamma(r) = \exp^\perp(rW)$ (in polar coordinates, $\gamma(r) = (r, 0)$). Along γ , f can be expressed as:

$$f(r, 0) = r \sqrt{F(rW, U, U)}.$$

The Gauss curvature of S at $(r, 0)$ equals $\frac{-f_{rr}(r, 0)}{f(r, 0)}$, where f_{rr} denotes the second partial with respect to r . The result now follows by performing the differentiation and taking the limit as $r \rightarrow 0$. □

It is now possible to express $K''_{W, X, V, Y, U}(0)$ in terms only of the connection, the warping tensor, and the metric of the soul:

Proposition 3.4.5.

$$\begin{aligned}
K''_{W, X, V, Y, U}(0) &= 2K(X, Y) + 6\langle R^\nabla(X, Y)U, V \rangle - 3\Upsilon(U, U, V, V) \\
&\quad + 2\langle (D_X R^\nabla)(X, Y)W, V \rangle + \frac{1}{2}|R^\nabla(W, V)X|^2 \\
&\quad - 2D_X \Upsilon(W, U, V, V) + 2D_X \Upsilon(W, V, U, V) - \frac{1}{2} \text{hess} \Upsilon(W, W, V, V)(X).
\end{aligned}$$

Proof. The formula is obtained from the formula of Proposition 3.2.3 by using Lemma 3.4.4 to re-writing the term $K(U, V)$ and using Lemma 3.4.3 to re-write the three T -tensor terms. □

Corollary 3.4.6. *For all $X, Y \in T_p M$ and all $V, W \in \nu_p(\Sigma)$:*

$$1. \quad 2\langle (D_X R^\nabla)(X, Y)W, V \rangle \leq \frac{1}{2}|R^\nabla(W, V)X|^2 + 2K(X, Y) - \frac{1}{2}\text{hess}_{\Upsilon(W, W, V, V)}(X).$$

Equivalently,

$$2. \quad \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (|R^\nabla(W, V)X|^2 - \text{hess}_{\Upsilon(W, W, V, V)}(X)) \cdot K(X, Y).$$

Proof. Part 1 is obtained by setting $U = 0$ in Proposition 3.4.5, and noting that $K''(0) \geq 0$. The equivalence of part 1 and part 2 is proven as in Corollary 3.2.4. \square

3.5 Warped connection metrics on vector bundles

In Section 3.3, we studied conditions for the existence of a connection metric of nonnegative curvature on a vector bundle. But connection metrics are a very special class of metrics. It might be possible for a bundle to admit a metric of nonnegative curvature, but fail to admit a connection metric of nonnegative curvature (deciding whether this phenomenon occurs remains an open problem).

In Section 3.4, we began to generalize away from connection metrics; specifically, Corollary 3.4.6 provides a necessary condition for the existence of an arbitrary metric of nonnegative curvature on a bundle (namely, the bundle must admit a connection ∇ and a tensor Υ which together satisfy the differential equation of the corollary). The goal of this section is to prove that this condition is almost sufficient, at least for the existence of a metric which is nonnegatively curved to second order. We begin by constructing warped connection metrics on vector bundles.

By a “warped connection metric” on a vector bundle $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$, we mean a metric which arises by choosing the following data:

1. A metric g_Σ on Σ .
2. A Euclidean structure on the bundle.
3. A connection ∇ on the bundle which is compatible with the Euclidean structure.
4. A smooth “warping function” F (for each $p \in \Sigma$, $F_p : E_p \times E_p \times E_p \rightarrow \mathbb{R}$) which satisfies:
 - (a) $F(0, U, V) = \langle U, V \rangle$.
 - (b) For fixed W , $F(W, \cdot, \cdot)$ is a symmetric positive-definite bilinear form.
 - (c) $F(W, W, U) = \langle W, U \rangle$.

Then there is a unique metric g_E on E (called a warped connection metric) for which $\pi : (E, g_E) \rightarrow (\Sigma, g_\Sigma)$ is a Riemannian submersion whose horizontal distribution is determined by ∇ and such that $\langle (d\exp^\perp)_W U, (d\exp^\perp)_W V \rangle = F(W, U, V)$ for every $W, U, V \in E_p$. To construct g_E explicitly, one can begin with the connection metric with flat fibers on the bundle, and then alter the metric on the vertical space according to the warping function F .

In other words, a warped connection metric is an arbitrary metric for which the bundle projection is a Riemannian submersion and $\exp^\perp : \nu(\Sigma) \rightarrow E$ is a diffeomorphism.

The following properties of F are consequences of the definition of a warping function (particularly of the smoothness of F):

- $\frac{d}{dt}|_{t=0} F(tW, U, V) = 0$
- $\frac{d^2}{dsdt}|_{s=t=0} F(tW_1 + sW_2, U, V) = \frac{d^2}{dsdt}|_{s=t=0} F(tU + sV, W_1, W_2).$

The warping tensor Υ corresponding to F is then defined as in the previous section, and has the properties of Lemma 3.4.2.

Warped connection metrics are much more general than connection metrics. In fact, it is clear from Section 3.4 that any metric of nonnegative curvature on a vector bundle agrees inside of the cut-locus of Σ with a warped connection metric:

Proposition 3.5.1. *Let M be an open manifold of nonnegative curvature with soul Σ . Then M agrees in a neighborhood Σ with a warped connection metric on its normal bundle.*

The following examples exhibit three simple classes of warped connection metrics.

Example 3.5.2 (Connection metrics). Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a function for which $G(0) = 0$, $G'(0) = 1$, and all even derivatives of G vanish at 0. Consider the following warping function:

$$F(W, U, V) = \frac{G^2(|W|)}{|W|^2} \langle U, V \rangle \quad \text{for all } U, V \perp W$$

(because of properties (b) and (c) in our definition of a warping function, F is completely determined by the values $F(W, U, V)$ when $U, V \perp W$). The corresponding warped connection metric on the bundle is just the connection metric whose radially-symmetric fiber metric is $ds^2 = dr^2 + G^2(r)d\Theta^2$. The corresponding warping tensor is completely determined by the following case:

$$\Upsilon(W, W, U, U) = \frac{2}{3}G'''(0) \cdot |W|^2 \cdot |U|^2 \quad \text{for all } W \perp U.$$

Example 3.5.3 (Simply-warped connection metrics). Let G_p ($p \in \Sigma$) be a smoothly varying family of functions such that for all $p \in \Sigma$, $G_p(0) = 0$, $G'_p(0) = 1$ and all even derivatives of G_p vanish at 0. For example, we could take:

$$G_p^2(r) := \frac{\epsilon^2(p)r^2}{\epsilon^2(p) + r^2}, \tag{3.5.1}$$

where $\epsilon : \Sigma \rightarrow \mathbb{R}$ is a smooth positively-valued function. Consider the following warping function:

$$F_p(W, U, V) = \frac{G_p^2(|W|)}{|W|^2} \langle U, V \rangle \quad \text{for all } U, V \perp W.$$

We call the corresponding metric a “simply-warped connection metric”. The fiber over $p \in \Sigma$ will have the radially symmetric metric $ds^2 = dr^2 + G_p^2(r)d\Theta^2$. The curvature of any vertical 2-plane at p is $-G_p'''(0)$. The corresponding warping tensor is determined by the following case:

$$\Upsilon_p(W, W, U, U) = \frac{2}{3}G_p'''(0) \cdot |W|^2 \cdot |U|^2 \quad \text{for all } W \perp U.$$

For example, equation 3.5.3.1 gives:

$$\Upsilon_p(W, W, U, U) = -\frac{2}{\epsilon^2(p)} \cdot |W|^2 \cdot |U|^2 \quad \text{for all } W \perp U. \tag{3.5.2}$$

Example 3.5.4 (Quadratically-warped connection metrics). Let Υ be a warping tensor, by which we mean a tensor of order 4 on the vector bundle which satisfies the following conditions:

1. $\Upsilon(W_1, W_2, U, V)$ is symmetric in (W_1, W_2) , and symmetric in (U, V) .
2. $\Upsilon(W, W, W, U) = 0$.
3. $\Upsilon(W_1, W_2, U_1, U_2) = \Upsilon(U_1, U_2, W_1, W_2)$

Consider the warping function:

$$F(W, U, V) := \langle U, V \rangle + \frac{1}{2} \Upsilon(W, W, U, V).$$

The corresponding metric will be called a “quadratically-warped connection metric”. The corresponding warping tensor is Υ . Notice that the metric in this example might not be positive-definite everywhere, but it is at least positive-definite in a neighborhood of the zero-section. For example, in the quadratically warped connection metric associated to the warping tensor defined by equation 3.5.3.2, the fiber E_p has the radially-symmetric metric $ds^2 = dr^2 + \bar{G}_p^2(r)d\Theta^2$ where $\bar{G}_p^2(r) = r^2 - \frac{1}{\epsilon^2(p)}r^4$, which is a well-defined metric for $r < \epsilon(p)$. Notice that $\bar{G}_p^2(r)$ is just the fourth degree Taylor polynomial of the function $G_p^2(r)$ defined by equation 3.5.3.1.

We list below some rigidity properties which warped connection metrics have in common with nonnegatively curved metrics on vector bundles:

Lemma 3.5.5. *Let $\pi : E \rightarrow \Sigma$ denote a vector bundle over Σ with a warped connection metric. Let F denote the warping function and let Υ denote the warping tensor. Let A, T denote the fundamental tensors of π . Let $p \in \Sigma$, $X, Y \in T_p\Sigma$ and let $U, V, W, W_1, W_2 \in E_p$. Then:*

1. *The zero-section, $\Sigma \subset E$, is totally geodesic.*
2. *The surface $(s, t) \mapsto \exp(s \cdot V(t))$ ($s, t \in \mathbb{R}, s \geq 0$), where $V(t)$ denotes the parallel transport of V along the geodesic with initial tangent vector X , is a flat and totally geodesic half-plane.*
3. *For sufficiently small r , the closed ball $\bar{B}_\Sigma(r)$ is convex.*
4. *$A_p = 0$ and $T_p = 0$.*
5. *$(D_V A)_X Y = \frac{1}{2} R^\nabla(X, Y) V$ and $(D_V A)_X U = -\frac{1}{2} R^\nabla(V, U) X$.*
6. *$D_W T = 0$ and $(D_{W_1} D_{W_2} T)_U V = -\frac{1}{2} \text{grad} \Upsilon(W_1, W_2, U, V)$.*
7. *$R(X, V) V = R(V, X) X = 0$ and $R(X, Y) U = 2R(X, U) Y$.*
8. *$K(X + U, Y + V) = K(X, Y) + K(U, V) + 3\langle X, Y, U, V \rangle$.*
9. *$K'_{W, X, V, Y, U}(0) = 0$.*
10. *$K''_{W, X, V, Y, U}(0)$ is given by the equation of Proposition 3.4.5.*

Proof. The only claim which is not immediate from previous arguments is that $D_W T = 0$, which is justified as follows. Let $\gamma(t) = \exp^\perp(tW)$. Then:

$$\begin{aligned} D_W T_U V &= \left. \frac{D}{dt} \right|_{t=0} (T_{\bar{U}} \bar{V})_{\gamma(t)} = -\frac{1}{2} \left. \frac{D}{dt} \right|_{t=0} (\overline{\text{grad} F(tW, U, V)})_{\gamma(t)} \\ &= -\frac{1}{2} \left. \frac{D}{dt} \right|_{t=0} \text{grad} F(tW, U, V) = -\frac{1}{2} \text{grad} \left(\left. \frac{d}{dt} \right|_{t=0} F(tW, U, V) \right) = 0 \end{aligned}$$

□

As a consequence of part 3 of Lemma 3.5.5, we get the following generalization of Proposition 3.3.5:

Proposition 3.5.6. *Suppose that $E \xrightarrow{\pi} \Sigma$ is a vector bundle with a warped connection metric g_E on E which is nonnegatively curved in a neighborhood of the zero-section $\Sigma \subset E$. Then there exists a complete metric g'_E on E which is everywhere nonnegatively curved.*

Theorem 3.5.7. *Suppose (Σ, g_Σ) is a compact Riemannian manifold with strictly positive sectional curvature. Suppose that $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$ is a vector bundle over Σ . Then there exists a warped connection metric g_E on E with soul (Σ, g_Σ) which has nonnegative curvature to second order iff the bundle admits a Euclidean structure, a connection ∇ , and a warping tensor Υ such that the following condition holds for all $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p$:*

$$(*) \quad \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (|R^\nabla(W, V)X|^2 - \text{hess}_{\Upsilon(W, W, V, V)}(X)) \cdot K(X, Y) \text{ with equality iff } Y \text{ is parallel to } X \text{ or } W \text{ is parallel to } V.$$

Lemma 3.5.8. *Condition $(*)$ of Theorem 3.5.7 is equivalent to any one of the following:*

- $(*\epsilon) \quad \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (1 - \epsilon) (|R^\nabla(W, V)X|^2 - \text{hess}_{\Upsilon(W, W, V, V)}(X)) \cdot K(X, Y)$ for some $\epsilon > 0$, with equality iff Y is parallel to X or W is parallel to V .
- $(**) \quad 2\langle (D_X R^\nabla)(X, Y)W, V \rangle \leq \frac{1}{2}(|R^\nabla(W, V)X|^2 - \text{hess}_{\Upsilon(W, W, V, V)}(X)) + 2K(X, Y)$. Further, when $X, W, V \neq 0$, there is equality iff Y is parallel to X and W is parallel to V .
- $(**\epsilon) \quad 2\langle (D_X R^\nabla)(X, Y)W, V \rangle \leq \frac{1}{2}(|R^\nabla(W, V)X|^2 - \text{hess}_{\Upsilon(W, W, V, V)}(X)) + 2(1 - \epsilon)K(X, Y)$ for some $\epsilon > 0$. Further, when $X, W, V \neq 0$, there is equality iff Y is parallel to X and W is parallel to V .

Proof. This is proven exactly like Lemmas 3.3.7 and 3.3.8. □

Lemma 3.5.9. *A warped connection metric g_E on the total space E of a vector bundle $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$ is nonnegatively curved to second order iff:*

1. *For every $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $U, V \in E_p$, $K(X + U, Y + V) \geq 0$ and $= 0$ iff X is parallel to Y and U is parallel to V .*
2. *If $p \in \Sigma$, $X, Y \in T_p \Sigma$, and $U, V, W \in E_p$ satisfy $|X| = |V| = |W| = 1$, $W \perp V$, $X \perp Y$, and $U \perp V$, then $K''_{W, X, V, Y, U}(0) > 0$.*

Proof. If g_E is nonnegatively curved to second order, then it satisfies hypothesis 1 and 2 of the lemma by definition. Conversely, assume that g_E satisfies hypotheses 1 and 2 of the lemma. Let $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $U, V, W \in E_p$ be arbitrary vectors with $W, V, X \neq 0$. If V is a multiple of W , then by Proposition 3.4.5, $K''_{W, X, V, Y, U}(0) = 2K(X + U, Y + V)$, which by property 1 of the lemma is ≥ 0 , and equals zero only when Y is parallel to X and U is parallel to V . So to prove that g_E has nonnegative curvature to second order, it remains to prove that if V is not a multiple of W , then $K''_{W, X, V, Y, U}(0) > 0$.

So assume that V is not a multiple of W . First, by Proposition 3.4.5, it is easy to see that

$$K''_{W, X, V, Y, U}(0) = K''_{W^\perp, X, V, Y^\perp, U^\perp}(0),$$

where W^\perp denotes the component of W which is orthogonal to V , Y^\perp denotes the component of Y which is orthogonal to X , and U^\perp denotes the component of U which is orthogonal to V .

Finally, we use the following identities, each of which can be easily checked either from the definition of $K''_{W, X, V, Y, U}(0)$ or from it's formula in Proposition 3.4.5:

- $K''_{aW, X, V, aY, aU}(0) = a^2 \cdot K''_{W, X, V, Y, U}(0)$.
- $K''_{W, aX, V, Y, aU}(0) = a^2 \cdot K''_{W, X, V, Y, U}(0)$.
- $K''_{W, X, aV, aY, U}(0) = a^2 \cdot K''_{W, X, V, Y, U}(0)$.

Using these properties, it is easy to choose λ_1, λ_2 such that:

$$K''_{W,X,V,Y,U}(0) = K''_{W^\perp,X,V,Y^\perp,U^\perp}(0) = K''_{\frac{W^\perp}{|W^\perp|}, \frac{X}{|X|}, \frac{V}{|V|}, \lambda_1 Y^\perp, \lambda_2 U^\perp}(0).$$

The last term of this equation is > 0 by hypothesis 2 of the lemma. \square

proof of Theorem 3.5.7. If the bundle admits a warped connection metric which is nonnegatively curved to second order, then condition $(**)$ is clearly satisfied. Conversely, suppose that condition $(**\epsilon)$ is satisfied. For fixed C , let Υ_C denote the warping tensor on the vector bundle which is determined by the following equation:

$$\Upsilon_C(W, W, U, U) = -\frac{2}{3} \cdot C \cdot |W|^2 \cdot |U|^2 \text{ for all } W \perp U.$$

In other words, Υ_C is the warping tensor associated to the connection metric for which the curvature of any vertical 2-plane at the zero-section is C (see Example 3.5.2). We will prove that for sufficiently large C , the quadratically-warped connection metric associated to the warping tensor $\tilde{\Upsilon} = \Upsilon + \Upsilon_C$ (see Example 3.5.4) is nonnegatively curved to second order.

By choosing C large, one can make the sectional curvature of all vertical 2-planes at Σ arbitrarily large. It is therefore easy to choose C sufficiently large that condition 1 of Lemma 3.5.9 is satisfied, exactly as done in the proof of Theorem 3.3.6. It remains to satisfy condition 2 of Lemma 3.5.9.

Let $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $W, U, V \in E_p$. Assume that $W \perp V$, $X \perp Y$, $U \perp V$, and $|W| = |X| = |V| = 1$. We must show that for sufficiently large C , $K''_{W,X,V,Y,U}(0) > 0$. Since Υ_C is parallel, $D_X \tilde{\Upsilon} = D_X \Upsilon$ and $\text{hess}_{\tilde{\Upsilon}} = \text{hess}_{\Upsilon}$. Hence, Proposition 3.4.5 applied to the metric associated to $\tilde{\Upsilon}$, gives the following formula:

$$\begin{aligned} K''_{W,X,V,Y,U}(0) &= 2\epsilon K(X, Y) + 6\langle R^\nabla(X, Y)U, V \rangle - 3\tilde{\Upsilon}(U, U, V, V) \\ &\quad - 2D_X \Upsilon(W, U, V, V) + 2D_X \Upsilon(W, V, U, V) \\ &\quad + 2(1 - \epsilon)K(X, Y) - \frac{1}{2}\text{hess}_{\Upsilon(W,W,V,V)}(X) \\ &\quad + 2\langle (D_X R^\nabla)(X, Y)W, V \rangle + \frac{1}{2}|R^\nabla(W, V)X|^2. \end{aligned} \tag{3.5.3}$$

Condition $(**\epsilon)$ says that the sum of the terms on the last two lines of equation 3.5.3.3 is ≥ 0 . Further, setting $Y = 0$ in $(**)$ and then using a compactness argument gives,

$$\frac{1}{2}|R^\nabla(W, V)X|^2 - \frac{1}{2}\text{hess}_{\Upsilon(W,W,V,V)}(X) > \delta > 0,$$

where δ depends on ∇ , Υ , and the metric g_Σ on Σ . It follows that there exist constants $\delta_1, \delta_2 > 0$ (depending on ∇ , Υ , g_Σ , and the Euclidean structure on the bundle) such that if $|Y| \leq \delta_1$ then the sum of the terms on the last two lines of equation 3.5.3.3 is $> \delta_2$.

Next let H denote the sum of the terms on the first two lines of equation 3.5.3.3. Since $-3\tilde{\Upsilon}(U, U, V, V) = -3\Upsilon(U, U, V, V) + 2C \cdot |U|^2$, we have:

$$\begin{aligned} H &= 2\epsilon K(X, Y) + 6\langle R^\nabla(X, Y)U, V \rangle - 3\Upsilon(U, U, V, V) + 2C \cdot |U|^2 \\ &\quad - 2D_X \Upsilon(W, U, V, V) + 2D_X \Upsilon(W, V, U, V). \end{aligned}$$

To complete the proof, it will suffice show that for sufficiently large C :

1. if $|Y| \leq \delta_1$ then $H + \delta_2 \geq 0$, and
2. if $|Y| > \delta_1$ then $H > 0$.

It is straightforward to choose C (depending on δ_1, δ_2 , the minimal sectional curvature of Σ , and the norms of the tensors R^∇ , Υ , and $D_X \Upsilon$) such that these two conditions are met. \square

Example 3.5.10. If the warping tensor Υ in Theorem 3.5.7 can be taken to be of the simple form:

$$\Upsilon(W, W, U, U) = -C \cdot |W|^2 \cdot |U|^2 \text{ for all } W \perp U,$$

then the resulting metric will be a connection metric (see Example 3.5.2). In this case, Theorem 3.5.7 reduces to Theorem 3.3.6.

Corollary 3.5.11. *Suppose (Σ, g_Σ) is a compact Riemannian manifold with strictly positive sectional curvature. Suppose that $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$ is a vector bundle over Σ . Then there exists a warped connection metric g_E on E with soul (Σ, g_Σ) which has nonnegative curvature to second order and radially-symmetric fibers iff the bundle admits a Euclidean structure, a connection ∇ , and positive real valued function $f : \Sigma \rightarrow \mathbb{R}$ such that the following condition holds for all $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p$:*

$$(*) \quad \langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (|R^\nabla(W, V)X|^2 + |W \wedge V|^2 \cdot \text{hess}_f(X)) \cdot K(X, Y) \text{ with equality iff } Y \text{ is parallel to } X \text{ or } W \text{ is parallel to } V.$$

Proof. If a warped connection metric has radially symmetric fibers, then the warping tensor has the form:

$$\Upsilon(W, W, U, U) = -f(p) \cdot |W|^2 \cdot |U|^2 \text{ for all } W \perp U,$$

where f is a positive real-valued function on Σ . Conversely, the quadratically-warped connection metric associated to a warping tensor of this form has radially-symmetric fibers. \square

3.6 Summary of conditions for nonnegative curvature

The following is a review of Theorem 3.3.6, Theorem 3.5.7 and Corollary 3.5.11:

Theorem 3.6.1 (Review of Results in this Chapter). *Suppose (Σ, g_Σ) is a compact Riemannian manifold with strictly positive sectional curvature. Suppose that $\mathbb{R}^k \rightarrow E \xrightarrow{\pi} \Sigma$ is a vector bundle over Σ .*

1. *There exists a connection metric g_E on E with soul (Σ, g_Σ) which has nonnegative curvature to second order iff the bundle admits a Euclidean structure and a connection ∇ such that the following condition holds for each $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p$:*

$$\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq |R^\nabla(W, V)X|^2 \cdot K(X, Y) \text{ with equality iff } Y \text{ is parallel to } X \text{ or } W \text{ is parallel to } V.$$

2. *There exists a warped connection metric g_E on E with soul (Σ, g_Σ) which has nonnegative curvature to second order and radially-symmetric fibers iff the bundle admits a Euclidean structure, a connection ∇ , and positive real valued function $f : \Sigma \rightarrow \mathbb{R}$ such that the following condition holds for all $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p$:*

$$\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (|R^\nabla(W, V)X|^2 + |W \wedge V|^2 \cdot \text{hess}_f(X)) \cdot K(X, Y) \text{ with equality iff } Y \text{ is parallel to } X \text{ or } W \text{ is parallel to } V.$$

3. *There exists a warped connection metric g_E on E with soul (Σ, g_Σ) which has nonnegative curvature to second order iff the bundle admits a Euclidean structure, a connection ∇ , and a warping tensor Υ such that the following condition holds for all $p \in \Sigma$, $X, Y \in T_p \Sigma$ and $V, W \in E_p$:*

$$\langle (D_X R^\nabla)(X, Y)W, V \rangle^2 \leq (|R^\nabla(W, V)X|^2 - \text{hess}_\Upsilon(W, W, V, V)(X)) \cdot K(X, Y) \text{ with equality iff } Y \text{ is parallel to } X \text{ or } W \text{ is parallel to } V.$$

In order to translate the above theorem into a necessary and sufficient condition for a bundle to admit a metric of nonnegative curvature (instead of a metric which is nonnegatively curved to second order), one must answer the following question:

Question 3.6.2. Is it true that a vector bundle admits a metric of nonnegative curvature iff it admits a metric which is nonnegatively curved to second order?

Neither direction is obvious. If a vector bundle admits a metric of nonnegative curvature, it is not obvious that it admits one for which $K''_{W,X,V,Y,U}(0) = 0$ only when X is parallel to Y and both U and V are parallel to W . The requirement that V is parallel to W seems especially strong; perhaps this hypothesis must be removed from the definition of “nonnegatively curved to second order” in order that Question 3.6.2 is true.

On the other hand, if a vector bundle admits a metric which is nonnegatively curved to second order, it seems computationally possible to check whether the quadratically-warped connection metric on the bundle associated to its warping tensor has nonnegative curvature in a neighborhood of the zero-section. The author is currently working on this question.

We end with an example, due to Walschap, which helps justify the need for our generalization from connection metrics to warped connection metrics.

Example 3.6.3. Consider the manifold $M = (S^2 \times \mathbb{R}^2) \times_{\mathbb{R}} \mathbb{R}$, where \mathbb{R} acts on S^2 by rotation about the north and south poles, on \mathbb{R}^2 by rotation, and on \mathbb{R} by translations. M is diffeomorphic to $S^2 \times \mathbb{R}^2$, but the induced submersion metric on M is a metric of nonnegative curvature which is different from the product metric. In particular, the fibers of the Sharafutdinov map are not totally geodesic, and the connection on the normal bundle of the soul is not a flat connection (see [39, page 529]). This connection is not capable of inducing a connection metric of nonnegative curvature on the bundle; this follows from [39, Theorem 1.5]. Thus, when a connection on a bundle is fixed, it is possible that this connection is capable of inducing a warped connection metric, but not a connection metric on the bundle.

Chapter 4

Riemannian submersions with compact holonomy

We say that a Riemannian submersion $\pi : M \rightarrow B$ has compact holonomy if its holonomy group, Φ , is a compact finite dimensional Lie group. We show by example that the metric projection onto a soul does not necessarily have compact holonomy. When the metric projection onto a soul does happen to have compact holonomy, we achieve the following splitting theorem: if the vertical curvatures of M decay to zero away from the soul, the M splits locally isometrically over its soul.

4.1 Consequences of compact holonomy

Our first consequence of compact holonomy is essentially due to Schroeder and Strake [27, Proposition 1]:

Lemma 4.1.1. *Let $\pi : M \rightarrow B$ be a Riemannian submersion. If π has compact holonomy, then there exists a constant $b_1 = b_1(\pi)$ such that the holonomy group is:*

$$\Phi = \{h^\alpha \mid \alpha \text{ is a piecewise smooth loop in } B \text{ at } p \text{ of length } \leq b_1\}.$$

In other words, the entire holonomy group can be represented by loops in B of bounded length. Since Schroeder and Strake stated this only for the case of the holonomy group of the normal bundle of the soul, we include here a translation of their proof into the above generality. As Schroeder and Strake point out, the proof is essentially a modification of [21, Appendix 4].

Proof. Fix $p \in B$. Consider a loop α_0 at p in B for which $h^{\alpha_0} = \text{id} \in \Phi$. For example, this occurs when α_0 is the trivial loop, but may occur for nontrivial loops as well. If α_t , $t \in (-\epsilon, \epsilon)$, is a variation of α_0 , then $V := \frac{\partial}{\partial t}|_{t=0} h^{\alpha_t} \in \mathcal{G}$, where \mathcal{G} denotes the Lie algebra of Φ .

Our first goal is to prove that every vector of \mathcal{G} occurs in this way. To establish this, let $\mathfrak{h} \subset \mathcal{G}$ denote the set of all vectors in \mathcal{G} which do occur in this way. We first argue that \mathfrak{h} is a Lie sub-algebra of \mathcal{G} . Let α_t^i ($i = 1, 2$) denote two families of loops of the type described above. Let $V^i := \frac{\partial}{\partial t}|_{t=0} h^{\alpha_t^i}$ be the corresponding elements of \mathfrak{h} . By re-parameterizing α_t^1 one can demonstrate that $\lambda V^1 \in \mathfrak{h}$ for any $\lambda \in \mathbb{R}$. By considering the concatenation $\alpha_t^1 \circ \alpha_t^2$, one can demonstrate that $V^1 + V^2 \in \mathfrak{h}$. Finally, if we let β_t denote that path for which $\beta_t(s^2) = (\alpha_t^1 \circ \alpha_t^2 \circ (\alpha_t^1)^{-1} \circ (\alpha_t^2)^{-1})(s)$, then $\frac{\partial}{\partial t}|_{t=0} h^{\beta_t} = [V^1, V^2] \in \mathfrak{h}$.

Since \mathfrak{h} is a Lie sub-algebra of \mathcal{G} , the distribution $g \mapsto (L_g)_*(\mathfrak{h})$ is involutive, where L_g denotes left translation by $g \in \Phi$. The maximal integral manifold through $\text{id} \in \Phi$, denoted H , is the Lie subgroup of Φ corresponding to \mathfrak{h} .

We wish to prove that $H = \Phi^0$. Take any $g \in \Phi^0$. Let α denote a loop in B at p for which $h^\alpha = g$, and let α_t ($t \in [0, 1]$) denote a nulhomotopy of $\alpha_0 = \alpha$. Let $\gamma(t) := h^{\alpha_1-t}$, which is a path in G from id to g . We show that for each $t_0 \in [0, 1]$, $\gamma'(t_0) \in (L_{\gamma(t_0)})_*(\mathfrak{h})$, and hence that $g \in H$. Consider the following family of loops:

$$\beta_t := (\alpha_{t_0})^{-1} \circ \alpha_{t_0+t}.$$

Since $h^{\beta_0} = \text{id}$, $V := \frac{\partial}{\partial t}|_{t=0} h^{\beta_t} \in \mathfrak{h}$. Further, $(L_{\gamma(t_0)})_*(V) = \gamma'(t_0)$ by construction. This completes our proof that $H = \Phi^0$.

Thus, we can choose a finite collection, $\alpha_t^1, \dots, \alpha_t^N$, of families of closed loops of the type described above, such that the corresponding vectors, V^1, \dots, V^N , form a basis of \mathcal{G} .

Consider the map $f : (-\epsilon, \epsilon)^N \rightarrow G$ defined by:

$$f(t_1, \dots, t_N) := h^{\alpha_{t_1}^1 \circ \dots \circ \alpha_{t_N}^N}.$$

By construction f is nonsingular at $(0, \dots, 0)$, and therefore $f([-\frac{\epsilon}{2}, \frac{\epsilon}{2}]^N)$ contains a closed neighborhood U of the identity element of Φ . By compactness, the lengths of the loops $\alpha_{t_1}^1 \circ \dots \circ \alpha_{t_N}^N$ are bounded when $|t_i| \leq \frac{\epsilon}{2}$. Finally, since Φ is compact, it can be covered by finitely many left-translates of U . In other words, there is a finite collection of loops β_1, \dots, β_l at p in B such that any element $g \in \Phi$ can be written as:

$$g = h^{\beta_i} \cdot h^{\alpha_{t_1}^1 \circ \dots \circ \alpha_{t_N}^N} = h^{\beta_i \circ \alpha_{t_1}^1 \circ \dots \circ \alpha_{t_N}^N}$$

for some $i = 1, \dots, l$ and for some $t_i \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$. The statement of the proposition follows. \square

Our second consequence of compact holonomy is a global Lipschitz bound on all holonomy diffeomorphisms, at least when π has a bounded T -tensor. It is known that the holonomy diffeomorphism h^α satisfies a Lipschitz bound depending on the length of α . More precisely, Guijarro and Walschap proved the following in [17, Lemma 4.2]:

Lemma 4.1.2 (Guijarro-Walschap). *Let $\pi : M \rightarrow B$ denote a Riemannian submersion with bounded T -tensor ($|T| \leq C_T$). Let α be a path in B from p to q . Then the holonomy diffeomorphism $h^\alpha : F_p \rightarrow F_q$ satisfies the Lipschitz constant $e^{C_T \cdot \text{length}(\alpha)}$.*

Proof. Assume that α is parameterized proportional to arclength such that $\alpha(0) = p$ and $\alpha(1) = q$. Let $x \in F_p$ and let $V \in \mathcal{V}_x$ (=the tangent space to the fiber at x) be a unit-length vector. Let $b(s)$ ($s \in (-\epsilon, \epsilon)$) denote a path in F_p with $b'(0) = V$. For each s let $t \mapsto \bar{\alpha}_s(t)$ denote the horizontal lift of α with $\bar{\alpha}_s(0) = b(s)$. Let $V(t)$ denote the vector field along $\bar{\alpha}_0$ corresponding to this variation. $V(t)$ is an everywhere vertical vector field with $V(0) = V$ and $V(1) = (h^\alpha)_*(V)$. But it's easy to compute that $V'(t) = A_{\alpha'_0(t)}V(t) + T_{V(t)}\alpha'_0(t)$, so that if we let $f(t) = |V(t)|^2$, then

$$f'(t) \leq 2\langle V(t), T_{V(t)}\alpha'_0(t) \rangle \leq 2 \cdot C_T \cdot \text{length}(\alpha) \cdot f(t).$$

Thus $|V(1)|^2 = f(1) \leq e^{2C_T \text{length}(\alpha)}$. Taking the square root of both sides completes the proof. \square

We prove that in the case of compact holonomy, the holonomy diffeomorphisms satisfy a Lipschitz bound which does not depend on the length of the path. In the case of non-compact holonomy we at least find a Lipschitz bound which depends linearly, rather than exponentially, on the length of α :

Proposition 4.1.3.

1. *Let $\pi : M \rightarrow B$ be a Riemannian submersion with a bounded T -tensor ($|T| \leq C_T$). If π has compact holonomy, then there exists a constant $L = L(\pi)$ such that all holonomy diffeomorphisms of π satisfy the Lipschitz constant L .*

2. Even if Φ is non-compact, there exists a constant $\tilde{L} = \tilde{L}(C_T, \text{diam}(B))$ such that the holonomy diffeomorphism associated with any loop of length l satisfies the Lipschitz constant $1 + l \cdot \tilde{L}$.

Proof. For part 1, if c is a loop at p in B of arbitrary length, one can find a different loop \tilde{c} of length $\leq b_1$ which generates the same holonomy diffeomorphism: $h^c = h^{\tilde{c}}$, where b_1 is the constant from Lemma 4.1.1. It follows that if α is a path of arbitrary length in B between points p and q , then $h^\alpha = h^{\tilde{\alpha}}$ for a properly chosen path $\tilde{\alpha}$ of length $\leq \text{diam}(B) + b_1$. For example, take $\tilde{\alpha}$ equal to a minimal path β from p to q followed by a loop at q which generates the same holonomy diffeomorphism as $\beta^{-1} * \alpha$. Therefore $L := e^{C_T(\text{diam}(B) + b_1)}$ is as required for part 1 of the proposition.

For part 2, first notice that for any loop α of length $l \leq D := 2 \cdot \text{diam}(B) + 1$, h^α satisfies the Lipschitz constant $e^{C_T \cdot l} \leq 1 + l \cdot P$, where $P := C_T \cdot e^{C_T \cdot D}$ (that is, P equals the maximum derivative of the function $e^{C_T \cdot l}$ between $l = 0$ and $l = D$). But for a unit-speed loop α at p in B of arbitrary length l , it is possible to find \bar{l} loops, $\alpha_1, \dots, \alpha_{\bar{l}}$, each of length $\leq D$, such that $h^\alpha = h^{\alpha_1} \circ \dots \circ h^{\alpha_{\bar{l}}}$, where \bar{l} denotes the smallest integer which is $\geq l$. To do this, define α_i to be the composition of a minimal path in B from p to $\alpha(i-1)$, followed by $\alpha|_{[i-1, i]}$, followed by a minimal path from $\alpha(i)$ to p . Since for each i , h^{α_i} satisfies the Lipschitz constant $1 + P \cdot D$, it follows that h^α satisfies the following Lipschitz constant:

$$\bar{l}(1 + P \cdot D) \leq (l + 1)(1 + P \cdot D) = 1 + (1 + PD)l + PD \leq 1 + 2(1 + PD)l.$$

The final inequality holds whenever $l \geq 1$, but if $l < 1$ then h^α satisfies the Lipschitz constant $1 + (C_T \cdot e^{C_T \cdot 1})l$ (as above). So the choice $\tilde{L} := \max\{2(1 + PD), C_T \cdot e^{C_T}\}$ is as required for part 2 of the proposition. \square

We do not know whether the holonomy diffeomorphisms of the metric projection onto a soul always obey a global Lipschitz bound (this question was first asked by Guijarro and Walschap). However, for general Riemannian submersions, we have the following:

Example 4.1.4. For a general Riemannian submersion with bounded T -tensor, the holonomy diffeomorphisms need not satisfy a global Lipschitz bound. For example, consider the projection $\pi : S^2 \times \mathbb{R}^2 \rightarrow S^2$, where S^2 has the round metric, \mathbb{R}^2 has the flat metric, and $S^2 \times \mathbb{R}^2$ has the product metric. Let X denote the vector field on S^2 whose flow is rotation about the axis through the north and south poles N, S . Let Y be a vector field on S^2 which is orthogonal to X and vanishes at N, S . Let $\frac{\partial}{\partial \theta}(v)$ ($v \in \mathbb{R}^2$) denote the radial vector field on \mathbb{R}^2 , and let $W(v) = \mu(|v|) \cdot \frac{\partial}{\partial \theta}(v)$, where μ is a smooth bump function with support $[1, 2]$. We can consider X, Y to be horizontal vector fields and W to be a vertical vector field on $S^2 \times \mathbb{R}^2$ in the obvious way. Define a 2-plane distribution \mathcal{H} on $S^2 \times \mathbb{R}^2$ as follows:

$$\mathcal{H}(p, v) = \text{span} \left\{ Y(p), \frac{X(p)}{|X(p)|} + |X(p)| \cdot W(v) \right\}.$$

This distribution clearly extends continuously to the fibers over N, S . There is a unique metric on $S^2 \times \mathbb{R}^2$ for which π becomes a Riemannian submersion with horizontal distribution \mathcal{H} such that the fibers are isometric to flat \mathbb{R}^2 . The holonomy group Φ is isomorphic to \mathbb{R} , and its action on the fiber $\pi^{-1}(N) = \mathbb{R}^2$ is simply the flow along the vector field W . It is easy to see that arbitrarily long loops in S^2 are necessary to achieve arbitrarily large time parameters for this flow, and hence arbitrarily bad Lipschitz bounds for the associated holonomy diffeomorphism. But since the T -tensors vanishes outside of a compact set in $S^2 \times \mathbb{R}^2$, it clearly has bounded norm.

4.2 Vertizontal curvature decay

In this section, we explore a consequence of compact holonomy for the metric projection onto a soul. The application is related to a splitting theorem of Guijarro and Petersen which states that

if the curvatures of all 2-planes on M decay towards zero away from the soul then the soul must be flat [16]. By O'Neill's formula, one can then conclude that the A-tensor of π vanishes, and hence that M splits locally isometrically over the soul (see Remark 3.1.2). Guijarro and Walschap have demonstrated that if M has the property that all holonomy diffeomorphism of π obey a global Lipschitz bound, then one only needs to know that the curvatures of all vertizontal 2-planes (that is, 2-planes spanned by a horizontal and a vertical vector) decay towards zero away from the soul in order to conclude that M splits locally isometrically over its soul [17, Theorem 4.3.2]. Thus, we have as a corollary to Proposition 4.1.3:

Corollary 4.2.1. *If Φ is compact and the curvatures of vertizontal 2-planes on M decay towards zero away from Σ , then M splits locally isometrically over Σ .*

We do not know whether this corollary is true even when Φ is noncompact. A more difficult question is whether the holonomy diffeomorphism of the metric projection onto a soul must satisfy a global Lipschitz bound, even when the holonomy group is non-compact.

4.3 A soul with noncompact normal holonomy

In this section we provide an example of a simply connected open manifold, M , of nonnegative curvature for which the metric projection onto the soul has noncompact holonomy. Consider the following action of the Lie group \mathbb{R} on $S^2 \times \mathbb{C}^2 \times \mathbb{R}$:

$$((\varphi, \theta), z_1, z_2, t_0) \xrightarrow{t} ((\varphi, \theta + t), e^{\pi i t} z_1, e^{\lambda \pi i t} z_2, t_0 - t).$$

Here (φ, θ) denotes spherical coordinates on S^2 , and λ denotes an irrational real number. The quotient, $(S^2 \times \mathbb{C}^2) \times_{\mathbb{R}} \mathbb{R} = (S^2 \times \mathbb{C}^2 \times \mathbb{R})/\mathbb{R}$, is diffeomorphic to $S^2 \times \mathbb{C}^2$, and this identification provides a new nonnegatively curved metric \hat{g} on $S^2 \times \mathbb{C}^2$ under which the quotient map $S^2 \times \mathbb{C}^2 \times \mathbb{R} \rightarrow (S^2 \times \mathbb{C}^2) \times_{\mathbb{R}} \mathbb{R} \approx S^2 \times \mathbb{C}^2$ becomes a Riemannian submersion.

Let g denote the product metric on $S^2 \times \mathbb{C}^2$, and let V denote the Killing vector field associated to the \mathbb{R} action on $S^2 \times \mathbb{C}^2$; namely, $V((\varphi, \theta), z_1, z_2) = (\hat{\theta}, iz_1, \lambda iz_2)$. According to [5, Example 2], the new metric \hat{g} on $S^2 \times \mathbb{C}^2$ is obtained from g simply by, at each point, rescaling the norms of vectors parallel to V by a factor of $1/(1 + |V|^2)^{1/2}$.

It is easy to see that the soul of $M := (S^2 \times \mathbb{C}^2, \hat{g})$ will still be the zero section, $\Sigma = S^2 \times (0, 0)$, and that the metric projection $\pi : M \rightarrow \Sigma$ will still be the projection $(q, z_1, z_2) \mapsto q$. It is straightforward to show that the horizontal distribution \mathcal{H} of π can be described as follows:

$$\mathcal{H}_{((\varphi, \theta), z_1, z_2)} = \text{span} \left\{ (\hat{\varphi}, 0, 0), \left(\hat{\theta}, \frac{|\hat{\theta}|^2}{1 + |\hat{\theta}|^2} iz_1, \frac{|\hat{\theta}|^2}{1 + |\hat{\theta}|^2} \lambda iz_2 \right) \right\}$$

The two vectors in this expression correspond to the horizontal lifts of the spherical coordinate vectors $\hat{\varphi}$ and $\hat{\theta}$ in $T_{(\varphi, \theta)} S^2$.

If $\sigma(t) = (\varphi(t), \theta(t))$, $t \in [0, 1]$, is a loop in S^2 based at $q := (\varphi(0), \theta(0))$, it follows that the horizontal lift $\bar{\sigma}$ of σ to the point (q, z_1, z_2) in M will end at the point $\bar{\sigma}(1) = (q, e^{\pi i t_0} z_1, e^{\lambda \pi i t_0} z_2)$ where $t_0 = \int_0^1 \frac{|\hat{\theta}|^2}{1 + |\hat{\theta}|^2} \theta'(t) dt$.

In particular, the set of points in $\pi^{-1}(q)$ which can be achieved as endpoints of horizontal lifts to (q, z_1, z_2) of loops in S^2 at q is exactly $\{(q, e^{\pi i t_0} z_1, e^{\lambda \pi i t_0} z_2) \mid t_0 \in \mathbb{R}\}$. It follows from this that the holonomy group Φ of the normal bundle of Σ is exactly:

$$\Phi := \left\{ \begin{pmatrix} e^{\pi i t_0} & 0 \\ 0 & e^{\lambda \pi i t_0} \end{pmatrix} \mid t_0 \in \mathbb{R} \right\},$$

which is isomorphic to \mathbb{R} .

We next wish to compute the ideal boundary $M(\infty)$ of $M = (S^2 \times \mathbb{C}^2, \hat{g})$. Let $p \in \Sigma$ be either the north or south pole; that is, one of the two zeros of the vector field V . The fiber $\pi^{-1}(p)$ is totally geodesic because it is a connected component of the fixed point set of the following isometry of M :

$$((\varphi, \theta), z_1, z_2) \mapsto ((\varphi, \theta + \pi/2), z_1, z_2).$$

Let S_p denote the sphere of unit-vectors normal to the soul at p . Think of $S_p = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. For any angle $\alpha \in [0, \pi/2]$, let $T_\alpha := \{(z_1, z_2) \in S_p \mid |z_1| = \cos(\alpha)\}$. This is the standard decomposition of the 3-sphere into a family of tori. Notice that each torus T_α is the closure of an orbit of the action of Φ on S_p . For any two rays γ_1, γ_2 from p in M with initial tangent vectors $V_1 \in T_{\alpha_1}$ and $V_2 \in T_{\alpha_2}$, the distance in $M(\infty)$ between the rays is easily seen to be $d_\infty([\gamma_1], [\gamma_2]) = |\alpha_1 - \alpha_2|$. In other words, the tori into which S_p decomposes collapse to single points in the ideal boundary. We conclude that $M(\infty) = [0, \pi/2]$.

4.4 Measuring size in the holonomy of a vector bundle

In this section we wish to show that, in a vector bundle with a connection, the “size” of the holonomy element associated to a loop can be controlled linearly in terms of the length of the loop. Later, in section 6.2, we will generalize this result to the holonomy group of a Riemannian submersion with bounded tensors. The main lemma of this section, which says that a nulhomotopy of a loop can always be found with derivatives controlled in terms of the length of the loop, is central not only to our vector bundle result, but also to all of the main results of Chapter 6.

Let $\mathbb{R}^k \rightarrow E \rightarrow B$ be an Riemannian vector bundle. Assume that B is compact and simply connected. Let ∇ be a connection which is compatible with the inner products on the fibers. Let R^∇ denote the curvature tensor of ∇ . Fix $p \in B$. Let Φ denote the holonomy group, and \mathcal{G} its Lie algebra. Notice that Φ is a Lie subgroup of the orthogonal group, since it acts naturally by isometries on the unit sphere E_p^1 of the fiber E_p at p . For $V \in \mathcal{G}$ and $w \in E_p^1$, denote by $V(w)$ the value at w of the vector field on E_p^1 associated to V . We call a left-invariant metric, m , on Φ *acceptable* if for all $V \in \mathcal{G}$, $|V|_m \leq \sup_{w \in E_p^1} |V(w)|$. Notice that any left invariant metric on Φ can be made acceptable by rescaling. For $g \in \Phi$, we define $|g|$ as the supremum over all acceptable metrics on Φ of the distance between g and id . This provides a natural notion of “size” in the holonomy group.

Proposition 4.4.1. *There is a constant $C = C(B)$ such that for any piecewise smooth loop α in B , $|P_\alpha| \leq C \cdot C_R \cdot \text{length}(\alpha)$, where C_R denotes a bound on $|R^\nabla|$.*

This proposition is particularly interesting when the holonomy group of the vector bundle is noncompact, in which case (as in the example of Section 4.3), arbitrarily long loops in the base space may be needed to represent the whole holonomy group.

The proof of this proposition turns on the following lemma, which will be of central importance to later results as well:

Lemma 4.4.2. *Let B be a compact and simply connected Riemannian manifold. There exists a constant $Q = Q(B)$ such that for any piecewise smooth loop $\alpha : [0, 1] \rightarrow B$ (parameterized proportional to arclength), there exists a piecewise smooth nulhomotopy $H : [0, 1] \times [0, 1] \rightarrow B$ of α (that is, $H(0, t) = p = \alpha(0)$, $H(1, t) = \alpha(t)$) for which the natural coordinate vector fields along the image of H are everywhere bounded in norm as follows: $|\frac{\partial}{\partial s} H| \leq Q$ and $|\frac{\partial}{\partial t} H| \leq Q \cdot \text{length}(\alpha)$. In particular, this implies that the area, $A(H)$, of the image of H is $\leq Q^2 \cdot \text{length}(\alpha)$.*

Proof. Let $\alpha : [0, 1] \rightarrow B$ be a piecewise smooth loop in B . Assume that α is parameterized proportional to arclength. Denote $p := \alpha(0) = \alpha(1)$, and $l := \text{length}(\alpha)$. To start, we will assume that $l \leq \frac{1}{2} \text{inj}(B)$, in which case it is easy to construct a nulhomotopy of α with derivative

information controlled linearly in terms of l . Since $l \leq \frac{1}{2}\text{inj}(B)$, α lifts to a loop $\tilde{\alpha} = \exp_p^{-1} \circ \alpha$ at 0 in $T_p B$. The natural nulhomotopy of $\tilde{\alpha}$ is $\tilde{H}(s, t) := s \cdot \tilde{\alpha}(t)$ ($s \in [0, 1], t \in [0, 1]$). Clearly $|\frac{\partial}{\partial t}\tilde{H}|, |\frac{\partial}{\partial s}\tilde{H}| \leq l$. Letting $H := \exp_p \circ \tilde{H}$, which is a piecewise smooth nulhomotopy of α , we see that $|\frac{\partial}{\partial t}H|, |\frac{\partial}{\partial s}H| \leq Q_1 \cdot l$ for a properly chosen $Q_1 = Q_1(B)$.

Next we assume only that $l \leq D := 2 \cdot \text{diam}(B) + 1$. More precisely, we seek a constant Q_2 such that for any piecewise smooth loop α in B at p of length $\leq D$, there exists a piecewise smooth nulhomotopy H of α , with $|\frac{\partial}{\partial t}H|, |\frac{\partial}{\partial s}H| \leq Q_2$.

Suppose no such Q_2 exists. Then there must be a sequence $\alpha_i : [0, 1] \rightarrow B$ of piecewise smooth loops, each with length $\leq D$, such that the minimal derivative bounds of piecewise smooth nulhomotopies of the loops α_i go to infinity. By restricting to a subsequence, we can assume that α_i converges in the sup norm; in particular α_i is a Cauchy sequence. For i, j large enough that $\epsilon := d_{\text{sup}}(\alpha_i, \alpha_j) \leq \frac{1}{2}\text{inj}(B)$, the natural piecewise smooth homotopy $H : [0, \epsilon] \times [0, 1] \rightarrow B$ between $H_0(t) = H(0, t) = \alpha_i(t)$ and $H_\epsilon(t) = H(\epsilon, t) = \alpha_j(t)$, which retracts along shortest geodesics between corresponding points of the two curves, is well defined. More precisely, define $H(s, t) = c_t(s)$, where c_t is the minimal path between $c_t(0) = \alpha_i(t)$ and $c_t(\epsilon) = \alpha_j(t)$, parameterized so as to have the constant speed $d(\alpha_i(t), \alpha_j(t))/\epsilon$. Clearly $|\frac{\partial}{\partial s}H| \leq 1$. Further, $|\frac{\partial}{\partial t}H| \leq K$ for an appropriate constant $K = K(B)$. To see this, notice that for fixed t , the vector field $J_t(s) = \frac{\partial}{\partial t}H(s, t)$ along the geodesic $c_t(s)$ is a Jacobi field because it is the variational vector field of the family of geodesics which defines the homotopy. $J_t(s)$ is determined by its endpoints $V_1 := J_t(0) = \alpha_i'(t)$ and $V_2 := J_t(\epsilon) = \alpha_j'(t)$, each of whose norm is $\leq D$. We can thus take K as the supremum (over all pairs of vectors $V_1 \in T_{p_1}B, V_2 \in T_{p_2}B$ with $d(p_1, p_2) \leq \frac{1}{2}\text{inj}(B)$ and $|V_1|, |V_2| \leq D$) of the maximal norm of the Jacobi field along the shortest geodesic between p_1 and p_2 with end values V_1 and V_2 .

It follows that any piecewise smooth nulhomotopy H_i of α_i can be extended to a piecewise smooth nulhomotopy H_j of α_j , with similar derivative bounds. More precisely, if $|\frac{\partial}{\partial s}H_i| \leq Q_s$ and $|\frac{\partial}{\partial t}H_i| \leq Q_t$, then $|\frac{\partial}{\partial s}H_j| \leq Q_s + E(\epsilon)$ and $|\frac{\partial}{\partial t}H_j| \leq \max\{Q_t, K\}$, where $\lim_{\epsilon \rightarrow 0} E(\epsilon) = 0$. This provides a contradiction. Therefore, such a constant Q_2 exists.

Finally, we handle the case where $l = \text{length}(\alpha)$ is arbitrary. It is possible to find \bar{l} loops, $\alpha_1, \dots, \alpha_{\bar{l}}$, each of length $\leq D$, such that $P_\alpha = P_{\alpha_1} \circ \dots \circ P_{\alpha_{\bar{l}}}$, where \bar{l} denotes the smallest integer which is $\geq l$. This is done exactly as in the proof of Lemma 4.1.3, by defining α_i to be the composition of a minimal path in B from p to $\alpha(i-1)$, followed by $\alpha|_{[i-1, i]}$, followed by a minimal path from $\alpha(i)$ to p . Let $\gamma : [0, 1] \rightarrow B$ denote the composition of the loops α_i , re-parameterized proportional to arclength. Notice that $\text{length}(\gamma) \leq \bar{l} \cdot D$. It is straightforward to define a piecewise smooth homotopy $H : [0, \frac{1}{2}] \times [0, 1] \rightarrow B$ between $H(0, t) = \alpha(t)$ and $H(\frac{1}{2}, t) = \gamma(t)$ with $|\frac{\partial}{\partial s}H| \leq 2 \cdot \text{diam}(B)$ and $|\frac{\partial}{\partial t}H| \leq \bar{l} \cdot D$. Next extend H by defining $H : [\frac{1}{2}, 1] \times [0, 1] \rightarrow B$ as the nulhomotopy of γ which simultaneously performs nulhomotopies of each loop α_i . H is clearly a piecewise smooth nulhomotopy of α for which $|\frac{\partial}{\partial s}H| \leq \max\{2 \cdot \text{diam}(B), 2Q_2\}$ and $|\frac{\partial}{\partial t}H| \leq \bar{l}Q_2 \leq (l+1)Q_2 \leq 2lQ_2$.

The final inequality above assumes that $l \geq 1$, but the case $l \leq 1$ can be handled as follows: If $l \leq \frac{1}{2}\text{inj}(B)$ then there exists a homotopy with $|\frac{\partial}{\partial s}H| \leq Q_1 \cdot l \leq Q_1$ and $|\frac{\partial}{\partial t}H| \leq Q_1 \cdot l$. On the other hand, if $\frac{1}{2}\text{inj}(B) \leq l \leq 1$ then there exists a homotopy with $|\frac{\partial}{\partial s}H| \leq \tilde{Q}_2$ and $|\frac{\partial}{\partial t}H| \leq \tilde{Q}_2 = \frac{\tilde{Q}_2 l}{l} \leq \frac{2\tilde{Q}_2}{\text{inj}(B)}l$. Here \tilde{Q}_2 is derived similarly to Q_2 , but for loops of length ≤ 1 rather than loops of length $\leq D$. In all cases, $|\frac{\partial}{\partial t}H|$ is bounded linearly in terms of l , and $|\frac{\partial}{\partial s}H|$ is bounded absolutely, which completes the proof. \square

Next we prove Proposition 4.4.1

Proof. Let $\alpha : [0, 1] \rightarrow B$ be a unit-speed piecewise smooth loop in B at p . By Lemma 4.4.2, there exists a piecewise smooth nulhomotopy $H : [0, 1] \times [0, 1] \rightarrow B$ of α such whose area $A(H)$ is bounded linearly in terms of $\text{length}(\alpha)$. We now describe how to control $|P_\alpha|$ linearly in terms of $A(H)$. Let $g(s) = P_{\{t \rightarrow H_s(t)\}}$, which is a piecewise smooth path in Φ between the identity and P_α .

For any vector w in E_p^1 , we define $w(s, t)$ as the parallel transport of w along $t \mapsto H_s(t)$. Then,

$$\begin{aligned}
|P_\alpha| &\leq \int_0^1 \sup\{|g'(s)|_m : m \text{ is acceptable}\} \, ds \\
&\leq \int_0^1 \sup_{w \in E_p^1} \left| \frac{D}{dS} \Big|_{S=s} g(S)(w) \right| \, ds \\
&= \int_0^1 \sup_{w \in E_p^1} \left| \frac{D}{dS} \Big|_{S=s} w(S, l) \right| \, ds \\
&\leq \int_0^1 \sup_{w \in E_p^1} \int_0^1 \left| \frac{D}{dt} \frac{D}{ds} w(s, t) \right| \, dt \, ds \\
&= \int_0^1 \sup_{w \in E_p^1} \int_0^1 \left| R^\nabla \left(\frac{\partial}{\partial t} H(s, t), \frac{\partial}{\partial s} H(s, t) \right) w(s, t) \right| \, dt \, ds \\
&\leq \int_0^1 \int_0^1 \sup_{w \in E_p^1} \left| R^\nabla \left(\frac{\partial}{\partial t} H(s, t), \frac{\partial}{\partial s} H(s, t) \right) w(s, t) \right| \, dt \, ds \\
&\leq |R^\nabla| \int_0^1 \int_0^1 \left| \frac{\partial}{\partial s} H(s, t) \wedge \frac{\partial}{\partial t} H(s, t) \right| \, dt \, ds \\
&= C_R A(H).
\end{aligned}$$

This completes the proof. □

Chapter 5

Volume growth bounds

The purpose of this chapter is to use Perelman's Theorem to study the volume growth, $\text{VG}(M)$, of M . In particular, $\text{VG}(M)$ is proven to be bounded above by the difference between the codimension of the soul and the maximal dimension, m , of an orbit of the action of Φ on $\nu_p(\Sigma)$. Additionally, if M satisfies an upper curvature bound, then we prove that $\text{VG}(M)$ is bounded below by one plus the dimension of $M(\infty)$. These two inequalities placed together read:

$$\dim(M(\infty)) + 1 \leq \text{VG}(M) \leq \text{codim}(\Sigma) - m. \quad (5.0.1)$$

It was first observed by Shioya in [30] that the first term of Equation 5.0.5.1 is \leq the last term (Shioya mistakenly defined m to be the dimension of Φ rather than the maximal dimension of an orbit of the action of Φ). Our result is therefore that $\text{VG}(M)$ is sandwiched between these two terms, at least when M satisfies an upper curvature bound.

5.1 Previous results

For a Riemannian manifold M^n with nonnegative sectional curvature, the volume of a ball $B_\Sigma(r)$ of radius r about a compact totally geodesic submanifold Σ^{n-k} can be estimated by a generalized Bishop-Gromov type inequality which states that

$$V(r) := \frac{\text{vol}(B_\Sigma(r))}{r^k}$$

is a monotonically non-increasing function of r [9]. If M is open, this implies that the volume growth of M is not greater than the codimension k of Σ , where the volume growth, $\text{VG}(M)$, of M is defined as follows:

$$\text{VG}(M) = \inf \left\{ x \in \mathbb{R} \mid \lim_{r \rightarrow \infty} \frac{\text{vol}(B_p(r))}{r^x} = 0 \right\}$$

Note that this definition is independent of the choice of $p \in M$.

The case where Σ is a soul of M and the volume growth of M is maximal in the sense that $\text{VG}(M) = k$ was studied by Schroeder and Strake in [27]. Their main result states that in this case the reduced holonomy group Φ^0 of the normal bundle $\nu(\Sigma)$ of Σ in M is trivial. Hence, in this case Theorem 2.2.1 implies that M splits locally isometrically over Σ , and the splitting is global if B is simply connected.

Our main theorem is the following generalization of Schroeder and Strake's result: $\text{VG}(M) \leq k - m$, where m denotes the maximal dimension of an orbit of the action of Φ^0 on $\nu_p(\Sigma)$. The possibility that Φ^0 might be a non-closed subgroup of $\text{SO}(k)$ introduces certain technical difficulties in our proofs. The previously constructed example in Section 4.3 justifies these worries.

It is worth mentioning that Perelman's theorem provides an elementary proof of the previously mentioned fact that $\text{VG}(M) \leq k$. For, if $\nu(\Sigma)$ is endowed with the connection metric with flat fibers, then \exp^\perp becomes a distance non-increasing map. To see this, take any $v \in \nu(\Sigma)$ and let $x := \exp^\perp(v) \in M$. Let $\tilde{\mathcal{H}}_v$ and $\tilde{\mathcal{V}}_v$ denote the horizontal and vertical spaces of $\nu(\Sigma)$ at v determined by the natural connection. Perelman's Theorem implies that $d(\exp^\perp)_v$ maps $\tilde{\mathcal{H}}_v$ isometrically onto \mathcal{H}_x . Also, Rauch's theorem implies that $d(\exp^\perp)_v$ maps $\tilde{\mathcal{V}}_v$ in a distance non-increasing way into \mathcal{V}_x . Since $\text{VG}(\nu(\Sigma)) = k$, it follows that $\text{VG}(M) \leq k$.

5.2 An upper bound on volume growth

In this section, we prove that $\text{VG}(M) \leq k - m$. Since Φ^0 might be non-compact, most of the theory of transformation groups does not apply to the action of Φ^0 on a fiber $\nu_p(\Sigma)$. We therefore require the following two lemmas about Lie subgroups of $\text{SO}(k)$.

Lemma 5.2.1. *Let G be a (possibly non-compact) connected Lie subgroup of $\text{SO}(k)$. The union, Ω , of all orbits of the action of G on $S^{k-1}(1)$ which have maximal dimension is open and dense.*

Proof. Ω is trivially open. To establish that Ω is dense, decompose the Lie algebra \mathcal{G} of G as $\mathcal{G} = \mathcal{Z} \oplus \mathcal{A}$, where \mathcal{Z} is the center of \mathcal{G} , and \mathcal{A} is semi-simple. Let Z and A denote the Lie subgroups of G associated with \mathcal{Z} and \mathcal{A} respectively. Z is Abelian and A is compact. Z is a subgroup of a maximal torus of $\text{SO}(k)$, and with respect to a properly chosen basis of \mathbb{R}^k , Z embeds in $\text{SO}(k)$ as follows:

$$Z := \left\{ \begin{pmatrix} e^{s_1(v)i} & & & \\ & \ddots & & \\ & & e^{s_m(v)i} & \\ & & & \text{I} \end{pmatrix} \mid v \in \mathcal{Z} \right\}$$

Here, each s_i is a linear function from \mathcal{Z} to \mathbb{R} . Write $\mathbb{R}^k = V_1 \oplus \dots \oplus V_m \oplus W$ for the corresponding orthogonal decomposition of \mathbb{R}^k ; $\dim(V_i) = 2$ and $\dim(W) = k - 2m$.

For any $v \in S^{k-1}(1)$, $\dim(G(v)) \leq \dim(Z(v)) + \dim(A(v))$, and we claim that equality holds on an open dense subset E of $S^{k-1}(1)$. To construct E , consider a basis $\{X_1, \dots, X_a\}$ of \mathcal{Z} and $\{Y_1, \dots, Y_b\}$ of \mathcal{A} , and regard the elements X_i and Y_i as Killing vector fields on \mathbb{R}^k . For each $i = 1, \dots, b$, the set $K_i := \{w \in W \mid Y_i(w) = 0\}$ is a linear subspace of W , which is not necessarily proper. Define:

$$E := \{v_1 + \dots + v_m + w \in S^{k-1}(1) \mid \text{each } v_i \neq 0 \text{ and if } w \in K_i \text{ then } K_i = W\}.$$

E is clearly open and dense. For every $v \in E$ and any $i = 1, \dots, b$, $Y_i(v) \notin \text{span}\{X_1(v), \dots, X_a(v)\}$. If i is such that $K_i \neq W$, this is clear from construction; otherwise, $Y_i(v) \in \text{span}\{X_1(v), \dots, X_a(v)\}$ implies $Y_i \in \mathcal{Z}$, which is impossible. Therefore, $\dim(G(v)) = \dim(Z(v)) + \dim(A(v))$ for every $v \in E$.

From the previous description of Z , one can easily see that the union, Ω_Z , of all orbits of the action of Z on $S^{k-1}(1)$ which have maximal dimension is open and dense in $S^{k-1}(1)$. Further, Ω_A (defined analogously) is open and dense by the theory of compact group actions. Therefore, $\Omega_Z \cap \Omega_A \cap E$ is an open dense subset of $S^{k-1}(1)$ which is contained in Ω . It follows that Ω is dense. \square

Lemma 5.2.2. *Let G be a (possibly non-compact) Lie subgroup of $\text{SO}(k)$, and C be a closed neighborhood of the identity in G . Let m denote the maximal dimension of an orbit of the action of G on \mathbb{R}^k . Then there exists a finite union, $W = \bigcup W_i$, of $(k - m)$ -dimensional subspaces W_i of \mathbb{R}^k such that for any $u \in \mathbb{R}^k$, there exists some $g \in C$ with $g(u) \in W$.*

Proof. The proof is by induction on k . The statement is trivial in dimensions $k = 1$ or 2 . Assume that the lemma is true in dimensions less than k . Let $C \subset G \subset \text{SO}(k)$, as in the statement of the lemma. For every $v \in S^{k-1}(1) \subset \mathbb{R}^k$, we will construct an open neighborhood U of v in $S^{k-1}(1)$ and a finite union, $W = \bigcup W_i$, of $(k - m)$ -dimensional subspaces W_i of \mathbb{R}^k which contain v , such that for every $u \in U$, there exists some $g \in C$ with $g(u) \in W$. Since $S^{k-1}(1)$ is compact, this will complete the proof.

Fix $v \in S^{k-1}(1)$. Let m_v denote the dimension of the orbit $G(v)$. Decompose the Lie algebra \mathcal{G} of G as $\mathcal{G} = \mathcal{G}_v \oplus \text{span}\{X_1, \dots, X_{m_v}\}$, where \mathcal{G}_v denotes the Lie algebra of the stabilizer G_v , and the elements X_i , when considered as Killing vector fields on $S^{k-1}(1)$, form a basis of $T_v G(v)$ at v . Define Y to be the orthogonal complement of $T_v G(v)$ in $T_v \mathbb{R}^k \approx \mathbb{R}^k$. Y is a $(k - m_v)$ -dimensional subspace of \mathbb{R}^k containing v , and we claim that the set $\{g(u) \mid g \in C, u \in S^{k-1}(1) \cap Y\}$ contains an open neighborhood U of v in $S^{k-1}(1)$. This follows from the fact that the derivative at (id, v) of the map $\Psi : G \times (Y \cap S^{k-1}(1)) \rightarrow S^{k-1}(1)$ defined as $\Psi(g, u) := g(u)$ is surjective.

If the orbit $G(v)$ has maximal dimension, that is if $m_v = m$, then the single subspace $W := Y$ and the open set U will be as required for the construction. Otherwise, the dimension of Y ($= k - m_v$) is too large, so Y cannot serve as the required subspace. In this case, we will choose W to be a union of smaller subspaces of Y , as follows.

The stabilizer G_v acts by isometries on the subspace Y' consisting of those vectors in Y which are orthogonal to v ; we claim that the maximal dimension of an orbit of this action is $m - m_v$. To see this, choose a neighborhood E of v in $Y \cap S^{k-1}(1)$ small enough so that $\{X_i(w)\}$ is linearly independent and transverse to Y for all $w \in E$. Then, for any $w \in E$, $G_v(w) \subset Y \cap S^{k-1}(1)$, and $T_w G(w) = T_w G_v(w) \oplus \text{span}\{X_i(w)\}$. Therefore, $\dim(G(w)) = \dim(G_v(w)) + m_v$. From lemma 5.2.1 and the fact that $G(E)$ contains an open neighborhood of v in $S^{k-1}(1)$, it follows that E must contain a vector w for which $\dim(G(w)) = m$. For this w , $\dim(G_v(w)) = m - m_v = \max\{\dim(G_v(w')) \mid w' \in Y'\}$.

By the inductive hypothesis, there exists a finite union, $V := \bigcup V_i$, of $(k - m - 1)$ -dimensional subspaces V_i of Y' such that for any $u \in Y'$ there exist an element $g \in G_v \cap C$ such that $g(u) \in V$. Define $W_i := \text{span}\{v, V_i\}$, and $W := \bigcup W_i$. Each subspace W_i has dimension $k - m$, and by construction, $\{g(u) \mid g \in G_v \cap C, u \in W\} = Y$. It follows that the set $\{g(u) \mid g \in C, u \in S^{k-1}(1) \cap W\}$ will contain the open neighborhood U of v defined above. This completes the proof. \square

Theorem 5.2.3. $VG(M) \leq k - m$

Proof. Fix $p \in \Sigma$. Lemma 4.1.1 states that if Φ^0 is compact, then all of Φ^0 can be represented using loops at p in Σ of bounded length. Even when the compactness assumption is dropped, the proof still establishes the following: there exist a constant b_1 and a closed neighborhood C of the identity in Φ^0 such that any $g \in C$ can be represented as parallel translation along a loop at p in Σ with length $\leq b_1$. By lemma 5.2.2, we can then find a finite union, $W := \bigcup W_i$, of $(k - m)$ -dimensional subspaces W_i of $\nu_p(\Sigma)$ which meets every orbit of C in the sense of the lemma.

Next we argue that all of M is contained in a ball of radius $b_2 := b_1 + \text{diam}(\Sigma)$ about $\exp^\perp(W)$. To see this, take any point $x \in M$, and write $x = \exp^\perp(v)$ for some $v \in \nu(\Sigma)$. There exists a path α in Σ of length less than b_2 such that $P_\alpha(v) \in W$ (namely, take any minimal path from $q := \pi(v)$ to p , followed by a loop at p which represents the proper holonomy element). The horizontal lift $\bar{\alpha}$ of α to x is then a path in M of the same length connecting x to $y := \exp^\perp(P_\alpha(v)) \in \exp^\perp(W)$. Therefore $B_{\exp^\perp(W)}(b_2) = M$. Additionally, observe that if v was chosen so that the radial geodesic $t \mapsto \exp^\perp(tv)$ provides a minimal connection from q to x , it follows that the radial geodesic $t \mapsto \exp^\perp(tP_\alpha(v))$ will provide a minimal connection from p to y .

We conclude with an argument similar to the proof of [27, Thm. 2], by showing that there is a constant K such that for large real numbers $r > 0$,

$$\text{Vol}(B_\Sigma(r) - B_\Sigma(r - 1)) \leq Kr^{k-m-1}$$

This clearly implies the statement of the theorem.

Let W^1 be the set of unit-length vectors in W . Let r be a positive real number. Take a net of points $\{v_i\}$ in W^1 such that every vector in W^1 makes an angle of less than $1/r$ with some v_i . Since W^1 is a finite union of round spheres of dimension $k - m - 1$, we need no more than $b_3 r^{k-m-1}$ points v_i , where b_3 is a constant which does not depend on r .

Now we claim that

$$B_\Sigma(r) - B_\Sigma(r-1) \subset \bigcup_i B_{\exp^\perp(rv_i)}(b_2 + 2).$$

To see this, take any point $x \in M$ with $r-1 \leq \text{dist}(x, \Sigma) \leq r$. As described above, x has distance $\leq b_2$ from some point $y = \exp^\perp(v)$, where $v \in W$, and $\gamma(t) := \exp^\perp(tv)$ provides a minimal connection between p and y . Let v_i be a point of the net which makes angle less than $1/r$ with v . Since γ is minimizing, we can apply Toponogov to conclude that $d(y, \exp^\perp(rv_i)) \leq 2$. Thus $\text{dist}(x, \exp^\perp(rv_i)) \leq b_2 + 2$.

Finally, since the Bishop-Gromov inequality implies that the volume of each ball $B_{\exp^\perp(tv_i)}(b_2 + 2)$ is not greater than the volume b_4 of a ball of the same radius in the Euclidean space of the same dimension, we have

$$\text{Vol}(B_\Sigma(r) - B_\Sigma(r-1)) \leq b_4 b_3 r^{k-m-1}.$$

So choosing $K := b_4 b_3$ is as required to conclude the proof. \square

5.3 Bounded vertical Jacobi fields

In this section we prove the existence of many bounded vertical Jacobi fields along radial geodesics (that is, geodesics which intersect the soul orthogonally). Since the existence of bounded vertical Jacobi fields restricts volume growth, this has the same flavor as Theorem 5.2.3.

Suppose that $p \in \Sigma$, $V \in \nu_p(\Sigma)$, and $\gamma(t) := \exp^\perp(tV)$. Choose vectors $X, Y \in T_p \Sigma$, and let $J(t)$ be the Jacobi field along γ with $J(0) = 0$ and $J'(0) = R^\nabla(X, Y)V$. Lemma 3.1.1 implies that $J(t)$ is an everywhere vertical Jacobi field which is bounded in norm because the A -tensor of π is bounded in norm. In other words, along γ we can find a number of linearly independent bounded vertical Jacobi fields equal to the dimension of the subspace $\{R^\nabla(X, Y)V \mid X, Y \in T_p \Sigma\}$. In fact, we can do slightly better:

Proposition 5.3.1. *Along almost every radial geodesic in M , there exist at least m linearly independent bounded vertical Jacobi fields, where m denotes the maximal dimension of an orbit of the action of Φ on $\nu_p(\Sigma)$.*

Proof. Begin by choosing a collection $\{\tau_1, \dots, \tau_a\}$ of paths in Σ beginning at p , and collections $\{X_1, \dots, X_a\}$, and $\{Y_1, \dots, Y_a\}$ of vectors in $T_p \Sigma$ such that the set $\{V_i := P_{\tau_i}^{-1} \circ R(P_{\tau_i}(X_i), P_{\tau_i}(Y_i)) \circ P_{\tau_i}\}$ is a basis of the Lie algebra of Φ^0 . This is possible by a theorem of Ambrose and Singer; see for example [21, Thm. 8.1]. Consider each V_i to be a vector field on the unit sphere $\nu_p^1(\Sigma)$ of $\nu_p(\Sigma)$. Let Ω be the union of all orbits of the action of Φ^0 on $\nu_p^1(\Sigma)$ which have maximal dimension. By Lemma 5.2.1, Ω has full measure. Fix $w \in \Omega$. $\{V_i(w)\}$ spans an m -dimensional subspace of $T_w(\nu_p^1(\Sigma))$. To simplify notation, re-index so that $\{V_1(w), \dots, V_m(w)\}$ is linearly independent. For each i between 1 and m , consider the Jacobi field $J_i(t)$ along $\gamma(t) := \exp^\perp(tw)$ with initial conditions $J_i(0) = 0$ and $J_i'(0) = V_i(w)$. We claim that each J_i is a bounded vertical Jacobi field.

To see this, let $\tilde{J}_i(t)$ be the Jacobi field along $\tilde{\gamma}_i(t) := \exp^\perp(tP_{\tau_i}(w))$ with initial conditions $\tilde{J}_i(0) = 0$ and $\tilde{J}_i'(0) = P_{\tau_i}(V_i(w))$. Lemma 3.1.1 implies that \tilde{J}_i is a bounded vertical Jacobi field.

Next observe that $\tilde{J}_i = (h^{\tau_i})_* J_i$, where $h^{\tau_i} : \pi^{-1}(p) \rightarrow \pi^{-1}(\tau_i(1))$ is the diffeomorphism between the fibers naturally associated to the path τ_i . This observation is a re-wording of [17, Prop. 1.1.1]. Also, each h^{τ_i} is a biLipschitz map according to Lemma 4.1.2. It follows that J_i is also a bounded vertical Jacobi field. \square

It would be interesting to find an alternative proof of Theorem 5.2.3 which is based on Proposition 5.3.1.

5.4 A lower bound on volume growth

The only known lower bound for the volume growth of an open manifold M of nonnegative curvature is due to Calabi, who proved that $VG(M) \geq 1$. Calabi's result appeared before Cheeger and Gromoll's Soul Theorem. Today the technology is in place to say much more. In this section we prove the following result:

Theorem 5.4.1. *If M satisfies an upper curvature bound, then the $VG(M) \geq 1 + \dim(M(\infty))$.*

We believe that this theorem is true even without the assumption that M satisfies an upper curvature bound. See [8] for an example of an open manifold of nonnegative curvature which does not satisfy an upper curvature bound.

We begin by stating a useful version of Toponogov's theorem, which can be found in [32, Lemma 19]:

Lemma 5.4.2 (Toponogov). *Let M be a Riemannian manifold of nonnegative curvature. Let $\alpha_i : [0, l_i] \rightarrow M$ ($i = 1, 2$) be two distance minimizing geodesics starting at the same point. Let $\theta(t_1, t_2)$ denote the "comparison angle"; that is, the angle opposite the third side of the triangle in flat \mathbb{R}^2 with side lengths $t_1, t_2, d_M(\alpha_1(t_1), \alpha_2(t_2))$. Then θ is monotonic in t_1 and t_2 . That is $\theta(t_1, t_2) \leq \theta(s_1, s_2)$ whenever $s_1 \leq t_1 \leq l_1$ and $s_2 \leq t_2 \leq l_2$.*

Corollary 5.4.3. *Suppose that α_i ($i = 1, 2$) are two unit speed rays from the same point and that a, b are positive real numbers. Define $g(t) := \frac{d_M(\alpha_1(at), \alpha_2(bt))}{t}$. Then $g(t)$ is monotonically decreasing in t , and*

$$\lim_{t \rightarrow \infty} g(t) = \sqrt{a^2 + b^2 - 2ab \cos \min\{d_\infty([\alpha_1], [\alpha_2]), \pi\}}. \quad (5.4.1)$$

Proof. The proof of Equation 5.4.5.1 is found in [31, Proposition 2.2]. The monotonicity claim follows immediately from Lemma 5.4.2 simply because $g(t)$ can be re-described as

$$g(t) = \sqrt{a^2 + b^2 - 2ab \cos \theta(at, bt)},$$

where θ is as in the lemma. □

The metric cone, $C(X)$, over a metric space X is defined as the quotient space $C(X) := X \times [0, \infty) / \sim$, where $(x, 0) \sim (y, 0)$ for any $x, y \in X$. The metric of $C(X)$ is defined by the law of cosines formula:

$$d((x, t), (y, s)) := \sqrt{s^2 + t^2 - 2st \cos \min\{d(x, y), \pi\}}.$$

Since $M(\infty)$ is an Alexandrov space with curvature bounded below by 1, it follows that $C(M(\infty))$ is an Alexandrov space with curvature bounded below by zero; see [4, Proposition 4.2.3]. The dimension of $C(M(\infty))$ is clearly one greater than the dimension of $M(\infty)$. A rephrasing of Theorem 5.4.1 is thus that the volume growth of M is bounded below by the dimension of $C(M(\infty))$; this is probably the best way to think about the claim. The following well known result, which provides motivation for Theorem 5.4.1, is due to Shiohama [29]:

Proposition 5.4.4 (Shiohama). *Let M be an open manifold of nonnegative curvature. Fix $p \in M$. Denote by λM the result of rescaling the metric on M by a factor of λ . Denote by o the vertex (cone point) of $C(M(\infty))$. Then the Gromov-Hausdorff limit (as λ goes to zero) of the pointed spaces $(\lambda M, p)$ is equal to $(C(M(\infty)), o)$*

Idea of the proof of Proposition 5.4.4. We sketch the proof found in [15, Lemma 3.4]. Let B denote the ball about o of radius 1. Fix $p \in M$. For any R , we wish to define a map $f_R : B \rightarrow B_p(R)$. An arbitrary point q of $C(M(\infty))$ can be denoted as $q = ([\gamma], t)$, where γ is a unit-speed ray in M from p and $t \in [0, \infty)$. The map f_R is then defined as follows:

$$f_R([\gamma], t) := \gamma(tR).$$

This definition uses implicitly the axiom of choice; for every point $q = ([\gamma], t)$ we are asked to randomly choose a ray γ from p representing the equivalence class $[\gamma]$. The map is thus non-canonical. Also it is not clear whether the choices can always be made so as to make f_R continuous. In any case, we can consider f_R as a map from B into the ball of radius 1 about p in $\frac{1}{R}M$. As $R \rightarrow \infty$, it follows essentially from Equation 5.4.5.1 that f_R becomes a better and better Hausdorff approximation (notice that the right hand side of Equation 5.4.5.1 can be interpreted as the distance in $C(M(\infty))$ between $([\alpha_1], a)$ and $([\alpha_2], b)$). The analogous construction holds when B is defined to be a ball of arbitrary radius in $C(M(\infty))$, and the proposition follows. \square

Although we did not need this observation in the previous proof, we mention now that f_R (interpreted as a map from B to the ball of radius 1 about p in $\frac{1}{R}M$) is distance non-decreasing. This follows from the monotonicity claim of Corollary 5.4.3. This observation is central to the following proof.

Proof of Theorem 5.4.1. Fix $p \in M$. Let B denote the ball of radius 1 about the vertex $o \in C(M(\infty))$ and let $f_R : B \rightarrow B_p(R)$ be defined as in the proof of Proposition 5.4.4. Let a denote any positive real number smaller than the dimension of $C(M(\infty))$. By the definition of the dimension of an Alexandrov space ([4, Section 6]),

$$\lim_{\epsilon \rightarrow 0} \epsilon^a \cdot \text{cap}_\epsilon(B) = \infty,$$

where $\text{cap}_\epsilon(B)$ denotes the maximal number of disjoint ϵ -balls which can be packed into B .

As mentioned previously, $f_R : B \rightarrow \frac{1}{R}B_p(R)$ is distance non-decreasing. This means that $d_M(f_R(q_1), f_R(q_2)) \geq R \cdot d(q_1, q_2)$. Therefore, any disjoint packing of $\frac{1}{R}$ -balls into B induces a disjoint packing of the same number of 1-balls into $B_p(R)$ via the map f_R .

Since M satisfies an upper curvature bound, it is shown in [44] that $\text{inj}(M) \geq C > 0$ for some C depending on the upper curvature bound and on the injectivity radius of the soul of M . Therefore it follows from a well known result of Croke that the volume of any ball of radius 1 on M is bounded below by a constant, \tilde{C} , depending on C and on the dimension of M [7].

Putting together the above observations,

$$\frac{\text{vol}(B_p(R))}{R^a} \geq \frac{\tilde{C} \cdot \text{cap}_{\frac{1}{R}}(B)}{R^a} \xrightarrow{R \rightarrow \infty} \infty,$$

which proves that $VG(M) \geq a$. This completes the proof. \square

Chapter 6

Bounded Riemannian submersions

In this chapter we generalize away from nonnegative curvature in order to study Riemannian submersion whose A and T tensors are both bounded. Since the metric projection onto a soul, $\pi : M \rightarrow \Sigma$, is an example of such a Riemannian submersion, the relevance of this study to nonnegative curvature is obvious. One consequence of the study will be that when the soul is simply connected, the ideal boundary of M can be determined completely from a single fiber of π . We also explore consequences outside of the field of nonnegative curvature; in particular, we prove that there are only finitely many isomorphism types among the class of Riemannian submersions whose base space and total spaces both satisfy fixed geometric bounds.

6.1 A bound on the folding of the fibers

Let $\pi : M^{n+k} \rightarrow B^n$ denote a Riemannian submersion. Assume that B is compact. Let A and T denote the fundamental tensors of π . Assume that $|A| \leq C_A$ and $|T| \leq C_T$. The main purpose of this chapter is to explore consequences of these bounds. For $p \in B$, denote by d_{F_p} the intrinsic distance function of the fiber $F_p = \pi^{-1}(p)$, and by d_M the distance function of M restricted to F_p . In this section we establish the following global metric property of the fibers of π :

Theorem 6.1.1. *If B is simply connected then:*

1. *There exists a constant $C_1 = C_1(B, C_A, C_T, k)$ such that for any $p \in B$, $d_{F_p} \leq C_1 \cdot d_M$.*
2. *If π has compact holonomy then there exists a constant $C_2 = C_2(\pi)$ such that for any two points $x, y \in F_p$ between which there exists a piecewise smooth horizontal path, $d_{F_p}(x, y) \leq C_2$.*

We begin by proving two lemmas which provide technical bounds on a Riemannian submersion with bounded tensors. Hereafter we denote by $Y^\mathcal{V}$ and $Y^\mathcal{H}$ the vertical and horizontal components of a vector $Y \in TM$.

Lemma 6.1.2.

1. *Along any horizontal path $\sigma(t)$ in M it is possible to construct an orthonormal vertical frame $\{\tilde{V}_1(t), \dots, \tilde{V}_k(t)\}$ with $|\tilde{V}_i'(t)| \leq 4^k \cdot k! \cdot C_A \cdot |\sigma'(t)|$.*
2. *If $Y(t)$ is any vector field along any horizontal path $\sigma(t)$ in M , then*

$$\frac{d}{dt}|Y(t)^\mathcal{V}| \leq k|(Y'(t))^\mathcal{V}| + k \cdot 4^k \cdot k! \cdot C_A \cdot |\sigma'(t)| \cdot |Y(t)|.$$

Proof. To establish part 1, let $\sigma(t)$ be a horizontal path in M . Let $\{Y_i(t)\}$, $i = 0..k$, denote the parallel transport along $\sigma(t)$ of an orthonormal basis $\{Y_i(0)\}$ of the *vertical* space at $\sigma(0)$. Denote by $Y_i(t) = X_i(t) + V_i(t)$ the decomposition of $Y_i(t)$ into horizontal and vertical components. Notice that

$$V_i'(t) = V_i'(t)^{\mathcal{H}} + V_i'(t)^{\mathcal{V}} = V_i'(t)^{\mathcal{H}} - X_i'(t)^{\mathcal{V}} = A(\sigma'(t), V(t)) - A(\sigma'(t), X(t)).$$

Therefore, $|V_i'(t)| \leq 2C_A|\sigma'(t)|$. We next define the frame $\{\tilde{V}_i(t)\}$ as the Gram-Schmidt orthonormalization of the (ordered) frame $\{V_i(t)\}$. For example, $\tilde{V}_1(t) = \langle V_1(t), V_1(t) \rangle^{-\frac{1}{2}} V_1(t)$. Differentiating this expression gives: $|\tilde{V}_1'(0)| \leq 2|V_1'(0)| \leq 4C_A|\sigma'(0)|$. Continuing the Gram-Schmidt process gives:

$$\tilde{V}_l(t) = \left(V_l(t) - \sum_{i=1}^{l-1} \langle \tilde{V}_i(t), V_l(t) \rangle \tilde{V}_i(t) \right)_{\text{normalized}}$$

Differentiating this expression gives:

$$\begin{aligned} |\tilde{V}_l'(0)| &\leq 2 \left(|V_l'(0)| + \sum_{i=1}^{l-1} (|V_l'(0)| + 2|\tilde{V}_i'(0)|) \right) \\ &\leq 2 \left(2lC_A|\sigma'(0)| + 2 \sum_{i=1}^{l-1} |\tilde{V}_i'(0)| \right) \leq a(l)C_A|\sigma'(0)|, \end{aligned}$$

where $a(l)$ is the solution to the following recurrence relation: $a(0) = 0$; $a(l) = 4 \left(l + \sum_{i=1}^{l-1} a(i) \right)$.

It is easy to see that $a(l) \leq 4^l \cdot l!$, which proves part 1 of the lemma.

Part 2 of the lemma follows by writing $|Y(t)^{\mathcal{V}}|^2 = \sum_{i=1}^k \langle Y(t), \tilde{V}_i(t) \rangle^2$ for the frame $\{\tilde{V}_i(t)\}$ given in part 1, and then differentiating with respect to t . \square

We use the previous Lemma to establish the following bound, which will be central to our proof of Theorem 6.1.1:

Lemma 6.1.3. *Let $\alpha_s(t) = \alpha(s, t)$ ($s \in [0, \epsilon], t \in [0, 1]$) denote a family of piecewise-smooth paths in B with fixed endpoints: $\alpha_s(0) = p, \alpha_s(1) = q$. Assume $|\alpha_0'(t)| \leq C_1$. Assume for the variational vector field $V(t) := \frac{\partial}{\partial s} \alpha(0, t)$ along α_0 that $|V(t)| \leq C_2$. Let $x \in F_p$. For each fixed s , let $t \mapsto \bar{\alpha}_s(t) = \bar{\alpha}(s, t)$ denote the horizontal lift the path $t \mapsto \alpha_s(t)$ with $\bar{\alpha}_s(0) = x$. Then $\tau(s) := \bar{\alpha}_s(1)$ is a path in the fiber F_q , and $|\tau'(0)| \leq \rho(1)$, where $\rho(t)$ denotes the solution to the following differential equation:*

$$\rho'(t) = kC_A C_1 C_2 (1 + 4^k k!) + kC_1 (C_T + 4^k k! C_A) \rho(t); \quad \rho(0) = 0. \quad (6.1.1)$$

Later, in section 7.3, we will be interested in the case $C_T = 0$. In anticipation of this we mention now that when $C_T = 0$, differential equation 6.1.6.1 simplifies greatly, and it is easy to show that:

$$\rho(1) \leq 2C_2 e^{k4^k(k+1)!C_1 C_A} \quad (6.1.2)$$

Proof. Let $\frac{\partial}{\partial t} \bar{\alpha}(s, t)$ and $\frac{\partial}{\partial s} \bar{\alpha}(s, t)$ denote the natural coordinate vector fields along the parameterized surface $\bar{\alpha}$. Notice that $\frac{\partial}{\partial t} \bar{\alpha}$ is everywhere horizontal. Also, $|\frac{\partial}{\partial t} \bar{\alpha}(0, t)| = |\frac{\partial}{\partial t} \alpha(0, t)| \leq C_1$ and $|\langle \frac{\partial}{\partial s} \bar{\alpha}(0, t), \bar{\alpha} \rangle^{\mathcal{H}}| = |\frac{\partial}{\partial s} \alpha(0, t)| \leq C_2$.

Applying part 2 of Lemma 6.1.2 to the vector field $\frac{\partial}{\partial s} \bar{\alpha}(0, t)$ along the horizontal curve $t \mapsto \bar{\alpha}(0, t)$ gives:

$$\begin{aligned}
\left| \frac{d}{dt} \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right| &\leq k \left| \left(\frac{D}{dt} \frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right| + k 4^k k! C_A C_1 \left| \frac{\partial}{\partial s} \bar{\alpha}(0, t) \right| \\
&= k \left| \left(\frac{D}{ds} \frac{\partial}{\partial t} \bar{\alpha}(0, t) \right)^\nu \right| + k 4^k k! C_A C_1 \left| \frac{\partial}{\partial s} \bar{\alpha}(0, t) \right| \\
&\leq k \left| A \left(\left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\mathcal{H}, \frac{\partial}{\partial t} \bar{\alpha}(0, t) \right) + T \left(\left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu, \frac{\partial}{\partial t} \bar{\alpha}(0, t) \right) \right| \\
&\quad + k 4^k k! C_A C_1 \left(\left| \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\mathcal{H} \right| + \left| \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right| \right) \\
&\leq k C_A C_1 C_2 + k C_T C_1 \left| \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right| \\
&\quad + k 4^k k! C_A C_1 \left(C_2 + \left| \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right| \right).
\end{aligned}$$

In other words,

$$\left| \frac{d}{dt} \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right| \leq k C_A C_1 C_2 (1 + 4^k k!) + k C_1 (C_T + 4^k k! C_A) \left| \left(\frac{\partial}{\partial s} \bar{\alpha}(0, t) \right)^\nu \right|.$$

Since $\tau'(0) = \frac{\partial}{\partial s} \bar{\alpha}(0, 1) = \left(\frac{\partial}{\partial s} \bar{\alpha}(0, 1) \right)^\nu$, this proves the lemma. \square

proof of Theorem 6.1.1. Let $x, y \in F_p$ and let $\gamma : [0, l] \rightarrow M$ be a shortest unit-speed path in M from x to y . Let \bar{l} denote the smallest integer which is not less than $l := \text{length}(\gamma)$. For each integer i between 0 and $\bar{l} - 1$, choose a point x_i of F_p closest to $\gamma(i)$, and choose a shortest unit-speed path τ_i from x_i to $\gamma(i)$. Next, for each integer i between 1 and \bar{l} , define γ_i as the concatenation of τ_{i-1} followed by $\gamma|_{[i-1, i]}$ followed by τ_i^{-1} . By construction, γ_i is a path in M between x_{i-1} and x_i , whose length, l_i , is not greater than $D := 2 \cdot \text{diam}(B) + 1$. We describe next how to construct a path β_i between x_{i-1} and x_i which remains in the fiber F_p , such that the length of β_i can be bounded linearly in terms of l_i .

Let $\alpha_i := \pi \circ \gamma_i$, which is a loop at p in B whose length is not greater than l_i . Let $z_i := h^{\alpha_i+1}(x_i)$. We first construct a path β_i^1 in F_p from x_{i-1} to z_{i-1} , and then construct a path β_i^2 in F_p from z_{i-1} to x_i .

To construct the path β_i^1 , first re-parameterize α_i proportional to arclength, so that $\alpha_i : [0, 1] \rightarrow B$. Find a piecewise smooth nulhomotopy $H : [0, 1] \times [0, 1] \rightarrow B$ of α_i . That is, $H(0, t) = p$, $H(1, t) = \alpha_i(t)$. By Lemma 4.4.2, H can be chosen so that $|\frac{\partial}{\partial t} H| \leq Q l_i$ and $|\frac{\partial}{\partial s} H| \leq Q$, where Q depends only on B . Lift the homotopy H to M by defining, for each $s \in [0, 1]$, the curve $t \mapsto \tilde{H}(s, t)$ to be the horizontal lift of the curve $t \mapsto H(s, t)$ beginning at $\tilde{H}(s, 0) = x_{i-1}$. Let $\beta_i^1(s) := \tilde{H}(s, 1)$. By Lemma 6.1.3, $|(\beta_i^1)'(s)| \leq \rho_{l_i}(1)$, where $\rho_{l_i}(t)$ is the solution to the following differential equation:

$$(\rho_{l_i})'(t) = k C_A Q^2 l_i (1 + 4^k k!) + k Q l_i (C_T + 4^k k! C_A) \rho_{l_i}(t); \quad \rho_{l_i}(0) = 0.$$

In particular, $\text{length}(\beta_i^1) \leq C \cdot l_i$, where C is a bound on the derivative of the function $l_i \mapsto \rho_{l_i}(1)$ between $l_i = 0$ and $l = D$.

We continue by constructing a path β_i^2 in F_p between z_{i-1} and x_i , whose length is also controlled linearly in terms of l_i . Let $\beta_i^2(s) := h^s(\gamma_i(s))$, where $h^s : F_{\alpha_i(s)} \rightarrow F_p$ is the holonomy diffeomorphism associated to the curve $\alpha_i|_{[s, l_i]}$. It is clear from construction that β_i^2 connects z_{i-1} to x_i and that $(\beta_i^2)'(s) = dh^s(\gamma_i'(s)^\nu)$. Thus, $\text{length}(\beta_i^2) \leq L \cdot l_i$, where L is a Lipschitz bound on all holonomy diffeomorphisms associated to curves in B of length $\leq D$ (such a Lipschitz bound exists by Lemma 4.1.2).

The concatenation, β_i , of β_i^1 followed by β_i^2 satisfies $\text{length}(\beta_i) \leq (C + L)l_i \leq (C + L)D$. The concatenation of the paths β_i ($i = 1, \dots, \bar{l}$) is a path from x to y in F_p . Therefore,

$$d_{F_p}(x, y) \leq \bar{l}(C + L)D \leq (l + 1)(C + L)D \leq 2l(C + L)D.$$

The final inequality is valid only when $l \geq 1$, but when $l \leq 1$, $d_{F_p}(x, y) \leq (C + L) \cdot l$. So the choice $C_1 := \max\{2(C + L)D, C + L\}$ is as required for part 1 of the theorem.

To prove part 2, let $x, y \in F_p$ be two points between which there exists horizontal path $\bar{\alpha}$. Let $\alpha := \pi \circ \bar{\alpha}$. Notice that $h^\alpha(x) = y$. One can choose a different loop σ at p in B such that $h^\sigma = h^\alpha$ and $\text{length}(\sigma) \leq b_1$, where b_1 is the constant from Lemma 4.1.1. The horizontal lift, $\bar{\sigma}$, of σ provides an alternative horizontal path from x to y . Thus $d_{F_p}(x, y) \leq C_1 \cdot d_M(x, y) \leq C_1 \cdot \text{length}(\sigma) \leq C_1 \cdot b_1$, so the choice $C_2 := C_1 b_1$ is as required for part 2. \square

Example 6.1.4. The conclusions of parts 1 and 2 Theorem 6.1.1 both fail for $\pi : TS^2 \rightarrow S^2$ when TS^2 is given the connection metric with flat fibers, which has unbounded A -tensor. It's easy to see that two rays from the origin of $T_p S^2$ grow apart linearly in the fiber $T_p S^2$, while they maintain a distance $\leq \text{length}(\alpha)$ in TS^2 , where α is a loop in S^2 which parallel translates the initial tangent vector of the first ray to the initial tangent vector of the second ray.

Example 6.1.5. The conclusions of parts 1 and 2 of Theorem 6.1.1 both fail for the following example in which the base space is not simple connected. Let M be the following flat manifold: $M = \mathbb{R}^2 \times [0, 1] / \{(r, \theta, 0) \sim (r, \theta + \lambda, 1)\}$. Here λ is any non-zero angle. Guijarro and Petersen studied this manifold in [16]. The soul of M is the circle $0 \times [0, 1] / \sim$, and each fiber F_p of the metric projection onto the soul is isometric to flat \mathbb{R}^2 . Two rays from the same point p of the soul which make an angle λ will grow apart linearly in the fiber F_p , but will remain at distance ≤ 1 in M .

Example 6.1.6. In section 4.3 we constructed a metric of nonnegative curvature on $S^2 \times \mathbb{R}^4$ for which the holonomy group of the metric projection onto the soul is noncompact. It is straightforward to see that the conclusion of part 2 of Theorem 6.1.1 fails for the metric projection onto the soul of this manifold.

Example 6.1.7. Let M^{n+k} be a simply connected open manifold of nonnegative curvature, and let $\pi : M^{n+k} \rightarrow \Sigma^n$ denote the metric projection onto its soul. Theorem 6.1.1 provides the bound $d_{F_p} \leq C \cdot d_M$ for each fiber F_p of π . According to the theorem, C depends on Σ, C_A, C_T , and k , where C_A and C_T denote bounds on the A and T -tensors of π respectively. But by O'Neill's formula (2.4.2.1), C_A depends only on the maximum curvature of Σ . Further, by Proposition 2.1.4, C_T depends only on the injectivity radius of Σ . In other words, C depends only on Σ and k . This means that our bound on the “folding of the fibers” holds uniformly over all open manifolds of nonnegative curvature of a fixed dimension and with a fixed soul!

6.2 Measuring size in the holonomy of a submersions

In Section 4.4 we proved that, for a vector bundle with a connection, the size of a holonomy element is bounded in terms of the length of the loop which generates it. In this section we generalize this result to bounded Riemannian submersions. Let $\pi : M \rightarrow B$ denote a Riemannian submersion whose fundamental tensors are bounded: $|A| \leq C_A$ and $|T| \leq C_T$. Assume that the holonomy group, Φ , of π is a (possibly noncompact) finite dimensional Lie group. We develop a notion of “size” in the holonomy group, and show how to control the size of h^α in terms of $\text{length}(\alpha)$. Let α_0 denote a loop in B at p for which $h^{\alpha_0} = \text{id}$. For example, this is the case when α_0 is the trivial loop, but it may also occur for nontrivial loops. Let α_s denote a variation of α_0 . Then h^{α_s} defines a path in Φ beginning at id , and $V := (h^{\alpha_s})'(0)$ is an element of the Lie algebra, \mathcal{G} , of Φ . Every vector of \mathcal{G} can be described in this way; see the proof of Lemma 4.1.1. It is natural to consider V as a vertical vector field on F_p . We write $V(x)$ for the value of this vector field at $x \in F_p$. It is clear from Lemma 6.1.3 that this vector field has bounded norm. This observation is particularly interesting for the metric projection onto a soul. In this setting, one can prove through other routes that such “holonomic” vertical vector fields are bounded; see for example Proposition 5.3.1.

We call a left invariant metric, m , on Φ *acceptable* if the following condition is satisfied: for all $V \in \mathcal{G}$, $|V|_m \leq \sup_{x \in F_p} |V(x)|$. Notice that any left invariant metric can be made acceptable by rescaling. For $g \in \Phi$, we denote by $|g|$ the supremum over all acceptable metrics on Φ of the distance in Φ between g and id . This provides a natural notion of the “size” of a holonomy element. Notice that $|g_1 \cdot g_2| \leq |g_1| + |g_2|$. We prove the following analog of Proposition 4.4.1:

Proposition 6.2.1. *If B is simply connected and Φ is a finite dimensional Lie group, then there exists $C = C(B, C_A, C_T, k)$ such that for any loop α in B , $|h^\alpha| \leq C \cdot \text{length}(\alpha)$.*

Proof. By an argument used many times in this paper, it will suffice to find a constant C such that $|h^\alpha| \leq C \cdot \text{length}(\alpha)$ for all loops α of length $\leq D = 2 \cdot \text{diam}(B) + 1$. Let $\alpha : [0, 1] \rightarrow B$ be a constant speed loop at $p \in B$ with $l := \text{length}(\alpha) \leq D$. By Lemma 4.4.2, there exists a piecewise smooth nulhomotopy $H : [0, 1] \times [0, 1] \rightarrow B$ of α such that $|\frac{\partial}{\partial t} H| \leq Ql$ and $|\frac{\partial}{\partial s} H| \leq Q$. Here $H_0(t) = H(0, t) = p$ and $H_1(t) = H(1, t) = \alpha(t)$. Let $g(s) = h^{H_s}$, which is a piecewise smooth path in Φ between the identity and h^α . It will suffice to find C such that for any acceptable metric, m , on Φ , $|g'(s)|_m \leq C \cdot l$. So choose any fixed acceptable metric, m . For fixed $s \in [0, 1]$, let σ_r be the following family of loops: $\sigma_r = H_{s+r} * (H_s)^{-1}$. Notice that $h^{\sigma_0} = \text{id}$, and $h^{H_s} \circ h^{\sigma_r} = h^{H_{s+r}}$. Let $V \in \mathcal{G}$ be the element $V := (h^{\sigma_r})'(0)$. Since the metric is left invariant, $|g'(s)|_m = |V|_m$. By Lemma 6.1.3, for any $x \in F_p$, $|V(x)| \leq \rho_l(1)$, where $\rho_l(t)$ denotes the solution to the following differential equation:

$$\rho'_l(t) = 2kC_A Q^2 l(1 + 4^k k!) + 2kQl(C_T + 4^k k! C_A) \rho_l(t) ; \quad \rho_l(0) = 0.$$

In particular, $|g'(s)|_m \leq C \cdot l$, where C is a bound on the derivative of the function $l \mapsto \rho_l(1)$ between $l = 0$ and $l = D$. This completes the proof. \square

We remark that Proposition 6.2.1 implies a weak version of Proposition 4.4.1. More precisely, if the unit sphere bundle of a Riemannian vector bundle with a connection is endowed with the natural connection metric, then the projection map becomes a Riemannian submersion. The T tensor of this submersion vanishes, and the A tensor depends on R^∇ . This argument produces only a weak version of Proposition 4.4.1, because it does not establish the bound to be linear in C_R .

6.3 The ideal boundary is determined by one fiber

We return in this section to our previous set up, in which M denotes an open manifold of nonnegative sectional curvature with soul $\Sigma \subset M$, and $\pi : M \rightarrow \Sigma$ denotes the metric projection. A nice way to rephrase the conclusion of Theorem 6.1.1 is that the inclusion map $i : F_p \hookrightarrow M$ is a biLipschitz map for any $p \in \Sigma$. Since any point of M has distance $\leq \text{diam}(\Sigma)$ from some point of the fiber F_p , i is a quasi-isometric embedding. In other words, M is quasi-isometric to any single fiber of the metric projection onto its soul.

From the above considerations, one expects a fiber F_p to share with M all large-scale geometric properties. In this section we prove that the ideal boundary of M is determined by a single fiber of π .

Corollary 6.3.1. *If $\pi_1(\Sigma) = 1$, then the topology of the ideal boundary, $M(\infty)$, of M can be determined by the pointed manifold (F_p, p) for any $p \in \Sigma$.*

Proof. Choose $p \in \Sigma$, and use the construction of $M(\infty)$ described in section 2.3. Any ray in F_p from p is also a ray in M , so we could equally well have defined $M(\infty)$ as the set of rays in F_p from p . By Theorem 6.1.1, we get the same topology on $M(\infty)$ if we replace d_M with d_{F_p} in equation 2.3.2.1. The statement of the corollary follows. \square

It does not make sense to speak of the “ideal boundary of a fiber”, since the term “ideal boundary” has only been defined for open manifolds of nonnegative curvature. However, speaking loosely, the above proof demonstrates that the ideal boundary of M is homeomorphic to the ideal boundary of any fiber.

Example 6.3.2. Corollary 6.3.1 fails for the open manifold of nonnegative curvature described in example 6.1.5 because the soul is not simply connected. In this example, $M(\infty) = S^1$ if the angle λ is rational, and $M(\infty)$ is a single point if λ is irrational. However, each fiber is isometric to flat \mathbb{R}^2 regardless of the value of λ .

Example 6.3.3. The manifold $M = (S^2 \times \mathbb{R}^4, \hat{g})$ constructed in Section 4.3 provides a nice illustration of Corollary 6.3.1. Here $M(\infty) = [0, \pi/2]$. There are two totally geodesic fibers of the metric projection $\pi : M \rightarrow \Sigma$; namely the fibers over the north and south poles of the soul $\Sigma = S^2 \times (0, 0)$. Each of these fibers is itself a manifold of nonnegative curvature whose ideal boundary also equals $[0, \pi/2]$. We do not know whether in general the ideal boundary of a totally geodesic fiber is necessarily isometric (rather than just homeomorphic) to the ideal boundary of the manifold.

We conclude this section by describing an alternative and substantially different way to prove (at least a weak version of) Theorem 6.1.1 in the special case of the metric projection onto a soul.

Proposition 6.3.4 (Special Case of Theorem 6.1.1). *Let M be a simply connected open manifold of nonnegative curvature, and let $\pi : M \rightarrow \Sigma$ denote the metric projection its soul. Then there exists a constant C depending on M such that for any two points x, y in the same fiber F_p which can be connected by a horizontal path γ in M , $d_{F_p}(x, y) \leq C \cdot \text{length}(\gamma)$.*

This proposition follows immediately from Theorem 6.1.1. In fact, since this proposition does not claim that C depends only on Σ , it is quite a bit weaker than Theorem 6.1.1 (see Example 6.1.7). Still, we wish to prove this proposition independently in order to exhibit a nice argument which is specific to nonnegative curvature. We begin with a lemma. Fix $p \in \Sigma$ and consider each vector V of the Lie algebra \mathcal{G} of the holonomy group Φ of $\nu(\Sigma)$ as a Killing vector field on the unit normal sphere $\nu_p^1(\Sigma)$. We write $V(w)$ for the value of this Killing field at $w \in \nu_p^1(\Sigma)$. Also, we fix an inner product on \mathcal{G} , and to consider Φ with the associated left-invariant metric.

Lemma 6.3.5. *There exists a constant K , depending only on M and the chosen inner product for \mathcal{G} , such that for any unit-length vector $V \in \mathcal{G}$ and any $w \in \nu_p^1(\Sigma)$, the Jacobi field $J(t)$ along the geodesic $t \mapsto \exp^\perp(tw)$ with $J(0) = 0$ and $J'(0) = V(w)$ satisfies $|J(t)| < K$ for all t .*

Proof. In the proof of Proposition 5.3.1, we constructed a basis $\{V_1, \dots, V_a\}$ of \mathcal{G} such that for any $w \in \nu_p^1(\Sigma)$ and any i between 1 and a , the Jacobi field $J_i(t)$ along $\gamma(t) := \exp^\perp(tw)$ with initial conditions $J_i(0) = 0$ and $J'_i(0) = V_i(w)$ is bounded in norm by a constant which does not depend on the choice of i or w . Since any unit-length vector $V \in \mathcal{G}$ can be written as $V = \sum \lambda_i V_i$, with each λ_i bounded in absolute value by a constant which reflects the extent to which the basis $\{V_i\}$ of \mathcal{G} fails to be orthonormal, the statement of the lemma follows. \square

Proof of Proposition 6.3.4. Let x, y be two points of the same fiber F_p between which there exists a horizontal path γ . Let $\alpha := \pi \circ \gamma$, which is a loop at p in Σ . Fix an inner product, m , on the Lie algebra \mathcal{G} of Φ which is acceptable in the sense of Section 4.4. Consider m as a left-invariant metric on Φ . By Proposition 4.4.1, there exists a constant \tilde{C} depending on Σ and on the connection of $\nu(\Sigma)$ such that

$$|P_\alpha|_m \leq \tilde{C} \cdot \text{length}(\alpha) \leq \tilde{C} \cdot \text{length}(\gamma).$$

This means that there is a vector $V \in \mathcal{G}$ of length $|V| \leq \tilde{C} \cdot \text{length}(\gamma)$ such that $\exp(V) = P_\alpha \in \Phi$.

Choose $w \in \nu_p^1(\Sigma)$ such that $x = \exp^\perp(rw)$, where $r := d_M(p, x)$. Define $\sigma(s) := \exp(s \cdot V)(w)$, which is a path in $\nu_p^1(\Sigma)$ with $\sigma'(s) = V(\sigma(s))$. Next define $\beta(s) := \exp^\perp(r \cdot \sigma(s))$, which is a path

in the fiber F_p between $\beta(0) = x$ and $\beta(1) = y$. Notice that $\beta'(s) = d(\exp^\perp)_{r \cdot \sigma(s)}(r \cdot \sigma'(s)) = J_s(r)$, where $J_s(t)$ is the Jacobi field along the radial geodesic $t \mapsto \exp^\perp(t \cdot \sigma(s))$ with $J(0) = 0$ and $J'(0) = V(\sigma(s))$. Therefore $|\beta'(s)| \leq K \cdot |V|$, where K is the constant given by Lemma 6.3.5. This means that $\text{length}(\beta) \leq K \cdot \tilde{C} \cdot \text{length}(\gamma)$. This completes the proof. \square

6.4 Two finiteness theorems for Riemannian submersions

In this section we prove two theorems which say that there are only finitely many equivalence classes of Riemannian submersions whose base space and total space both satisfy fixed geometric bounds. We consider two Riemannian submersions, $\pi_1 : M_1 \rightarrow B_1$ and $\pi_2 : M_2 \rightarrow B_2$, to be C^k -equivalent if there exists a C^k map $f : M_1 \rightarrow M_2$ which maps the fibers of π_1 to the fibers of π_2 . Every Riemannian submersion is a fiber bundle, and this notion of equivalence just means equivalence up to C^k fiber bundle isomorphism. Our first result is a finiteness theorem for C^1 -equivalence classes of submersions. Our second result will show that some of the geometric hypotheses of the first can be weakened if we are willing to settle for proving finiteness only up to C^0 -equivalence classes.

Theorem 6.4.1 (Finiteness Theorem #1). *Let $n, k \in \mathbb{Z}$ and $V, D, \lambda \in \mathbb{R}$. Then there are only finitely many C^1 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \rightarrow B^n$ for which:*

1. B is simply connected.
2. $\text{vol}(B) \geq V, \text{diam}(B) \leq D, |\text{sec}(B)| \leq \lambda$.
3. $\text{vol}(M) \geq V, \text{diam}(M) \leq D, |\text{sec}(M)| \leq \lambda$.

Some of the bounds in this theorem are redundant. For example, since Riemannian submersions are curvature non-decreasing, the lower curvature bound on the base space is redundant to the lower curvature bound on the total space. Also, the upper diameter bound on B is redundant to the upper diameter bound on M .

This theorem is based on the following result of P. Walczak ([36], as corrected in [37]):

Theorem 6.4.2 (Walczak). *Let $n, k \in \mathbb{Z}$ and $V, D, \lambda, C_A, C_T \in \mathbb{R}$. Then there are only finitely many C^1 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \rightarrow B^n$ for which:*

1. $|A| \leq C_A$ and $|T| \leq C_T$.
2. $\text{vol}(B) \geq V, \text{diam}(B) \leq D, |\text{sec}(B)| \leq \lambda$.
3. There exists a fiber F_p for which $\text{vol}(F_p) \geq V, \text{diam}(F_p) \leq D, |\text{sec}(F_p)| \leq \lambda$.

Proof of Theorem 6.4.1. Let $n, k \in \mathbb{Z}$ and let $V, D, \lambda \in \mathbb{R}$. Suppose that $\pi : M^{n+k} \rightarrow B^n$ is a Riemannian submersion satisfying conditions 1-3 of Theorem 6.4.1. By O'Neill's Formula (2.4.2.1), the A -tensor of π is bounded in norm by a constant, C_A , depending only on λ . Similarly, by Proposition 2.1.4, the T -tensor of π is bounded by a constant C_T which depends only on λ and on $\text{inj}(B)$. By a well known lemma of Cheeger, $\text{inj}(B)$ is in turn bounded below by a constant depending only on n, V, D , and λ (see [25]).

Let $p \in B$ and let $F_p := \pi^{-1}(p)$. It remains to bound the volume, diameter, and curvature of F_p in terms of $\{n, k, V, D, \lambda, C_A, C_T\}$, and then apply Theorem 6.4.2. First, by Gauss' formula, $|\text{sec}(F_p)| \leq \lambda + 2C_T$. Second, to control $\text{vol}(F_p)$, notice that any two fibers have similar volumes. More precisely, the diffeomorphism $h^\alpha : F_p \rightarrow F_q$ associated to a minimal path, α , in B between p and q satisfies the Lipschitz constant $e^{C_T \cdot \text{length}(\alpha)} \leq e^{C_T \cdot D}$ (see Lemma 4.1.2), so $\text{vol}(F_p) \leq (e^{kC_T \cdot D}) \cdot \text{vol}(F_q)$. But by Fubini's theorem, $\text{vol}(M) = \int_{p \in B} \text{vol}(F_p) d\text{vol}_B$, which implies that for

any $p \in B$, $\text{vol}(F_p) \geq \frac{\text{vol}(M)}{\text{vol}(B)}(e^{-kC_T \cdot D})$. By the Bishop-Gromov inequality, $\text{vol}(B)$ is bounded above by a constant depending only on D, λ , and n . This observation completes our argument that $\text{vol}(F_p)$ is bounded below.

Finally, $\text{diam}(F_p) \leq C_1 \cdot \text{diam}(M) \leq C_1 \cdot D$, where C_1 is the constant from Theorem 6.1.1, which depends on $\{B, C_A, C_T, k\}$. In fact, it is clear from the proof of theorem 6.1.1 that C_1 really depends only on $\{\text{diam}(B), Q(B), C_A, C_T, k\}$, where $Q(B)$ is the constant in Lemma 4.4.2. We argue now that $Q(B)$ depend only on the assumed geometric bounds of B . Let \mathfrak{M} denote the class of all n dimensional Riemannian manifolds for which $\text{vol} \geq V$, $\text{diam} \leq D$, and $|\text{sec}| \leq \lambda$. By assumption, $B \in \mathfrak{M}$. \mathfrak{M} is pre-compact in the Lipschitz topology and contains only finitely many diffeomorphism types (see [25]). This means that it is possible to choose a finite set, $\{B_1, \dots, B_l\} \subset \mathfrak{M}$, and a constant $L = L(n, V, D, \lambda)$ such that for any $M \in \mathfrak{M}$, there exists an L -biLipschitz diffeomorphism between M and some B_i . Therefore, for any $M \in \mathfrak{M}$, $Q(M) \leq L^2 \cdot \max\{Q(B_i)\}$. This proves that $Q(B)$ satisfies an upper bound depending only on $\{n, V, D, \lambda\}$, which completes the proof. \square

Next we discuss the following question: if the upper curvature bound on the base space and/or the total space is removed from Theorem 6.4.1, is it still true there are only finitely many C^0 fiber bundle isomorphism classes in this set of Riemannian submersions? We begin this exploration by recalling a Theorem of J.Y. Wu [43]:

Theorem 6.4.3 (J.Y. Wu). *Let B^n be a compact Riemannian manifold. Let $k \in \mathbb{Z}$ and $V, D, \lambda \in \mathbb{R}$. Assume $k \geq 4$. Then there are only finitely many C^0 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \rightarrow B^n$ for which:*

1. $\text{sec}(M) \geq \lambda$.
2. For each fiber F_p , $\text{vol}(F_p) \geq V$, $\text{diam}(F_p) \leq D$, $\text{sec}(F_p) \geq \lambda$.
3. Each fiber F_p is totally convex (that is, $d_{F_p} = d_M$).

The advantage of Wu's theorem is that he requires no upper curvature bounds. The disadvantages of Wu's theorem are as follows:

- The fiber dimension is restricted to $k \geq 4$. Wu stated in [43] that he believes the result should still hold when $k = 2, 3$.
- He fixes the base space rather than fixing geometric bounds on the base space.
- He assumes geometric bounds on the fibers rather than only assuming geometric bounds on the total space.
- The condition that the fibers are totally convex is very strong. For example, it is much stronger than assuming that the fibers are totally geodesic.

The following is a fairly immediate application of Theorem 6.1.1:

Theorem 6.4.4. *Wu's Theorem 6.4.3 remains true if hypothesis 3 is removed.*

Proof. It follows immediately from Wu's proof that hypothesis 3 can be replaced by the following weaker assumption:

- 3') For any fiber F_p , if $x, y \in F_p$ and $d_M(x, y) \leq \frac{1}{2}\text{inj}(B)$, then $d_{F_p}(x, y) \leq C \cdot d_M(x, y)$,

where C is any fixed constant.

Now suppose that $\pi : M \rightarrow B$ be a Riemannian submersion satisfying conditions 1 and 2 of Theorem 6.4.3. As before, the A -tensor is bounded in norm by a constant C_A depending only on B and λ , and the T -tensor is bounded in norm by a constant C_T depending only on $\text{inj}(B)$ and λ .

So if the base space B is simply connected then it is immediate from Theorem 6.1.1 that condition 3' is satisfied automatically for a constant C depending only on $\{B, V, D, \lambda, k\}$. In fact, this is true when B is not simply connected as well. To see this, notice that if $x, y \in F_p$ satisfy $d_M(x, y) \leq \frac{1}{2}\text{inj}(B)$, then the projection to B of a minimal path in M between x and y will be nulhomotopic, so the proof of Theorem 6.1.1 goes through in this setting. \square

Wu states in [43, p. 513] that his theorem remains true if “Riemannian submersion” is replaced by “ L -Lipschitz map”, where L is any fixed number. It is not clear to us that his proof goes through in this generality; however, it is obvious that the proof still holds when “Riemannian submersion” is replaced by “submersion π for which $\pi_*|_{\mathcal{H}}$ is at every point an L -biLipschitz linear map”, where \mathcal{H} denotes the distribution orthogonal to the fibers. We will call such a submersion an “ L -Riemannian submersion”. In review, the following generalization of Wu’s Theorem is true:

Theorem 6.4.5 (Generalization of Wu’s Theorem). *Let B^n be a compact Riemannian manifold. Let $k \in \mathbb{Z}$ and $V, D, \lambda, L \in \mathbb{R}$. Assume $k \geq 4$. Then there are only finitely many C^0 fiber bundle isomorphism classes in the set of L -Riemannian submersions $\pi : M^{n+k} \rightarrow B^n$ for which:*

1. $\text{sec}(M) \geq \lambda$.
2. For each fiber F_p , $\text{vol}(F_p) \geq V$, $\text{diam}(F_p) \leq D$, $\text{sec}(F_p) \geq \lambda$.

This jump to the generality of L -Riemannian submersions allows us to replace the fixed base space with a base space which only has fixed geometric bounds. We also replace the geometric bounds on the fibers with only geometric bounds on the total space:

Theorem 6.4.6 (Finiteness Theorem #2). *Let $n, k \in \mathbb{Z}$ and $V, D, \lambda \in \mathbb{R}$. Assume $k \geq 4$. Then there are only finitely many C^0 fiber bundle isomorphism classes in the set of Riemannian submersions $\pi : M^{n+k} \rightarrow B^n$ for which:*

1. B is simply connected.
2. $\text{vol}(B) \geq V$, $\text{diam}(B) \leq D$, $|\text{sec}(B)| \leq \lambda$.
3. $\text{vol}(M) \geq V$, $\text{diam}(M) \leq D$, $\text{sec}(M) \geq \lambda$.

Proof. Let \mathfrak{M} denote the set of all candidates for the base space (that is, all Riemannian n -manifolds satisfying $\text{vol} \geq V$, $\text{diam} \leq D$, and $|\text{sec}| \leq \lambda$). Choose $\{B_0, \dots, B_l\} \subset \mathfrak{M}$ and $L \in \mathbb{R}$ such that for each $B \in \mathfrak{M}$, there exists an L -biLipschitz diffeomorphism $f : B \rightarrow B_i$ for some i (as in the proof of Theorem 6.4.1).

Now if $\pi : M \rightarrow B$ is a Riemannian submersion satisfying the hypotheses of Theorem 6.4.6, then $f \circ \pi : M \rightarrow B_i$ is an L -Riemannian submersion. Further, the two submersions, π and $f \circ \pi$, are clearly isomorphic as fiber bundles. By arguments in the proof of Theorem 6.4.1, each fiber of π (equivalently of $f \circ \pi$) satisfies a curvature, diameter, and volume bound depending only on V, D, λ, n, k . Therefore, applying Theorem 6.4.5 to each of the base spaces B_0, \dots, B_l finishes the proof. \square

This theorem is probably not optimal. For example, the assumption that $k \geq 4$ can probably be removed, as well as the upper curvature bound on the base space. Also, we do not know whether the theorem holds without the assumption that the base space is simply connected. This hypothesis is used in the proof only to achieve an upper diameter bound for the fibers, so instead of assuming that the base space is simply connected, we could instead assume a diameter bound for the fibers.

Chapter 7

Finiteness theorems for fiber bundles

In this chapter we prove finiteness theorems for vector bundles and principal bundles. For example, there are only finitely many vector bundles of a fixed rank over a fixed compact Riemannian manifold capable of admitting a connection whose curvature tensor satisfies a fixed bound in norm. An analogous statement holds for principal bundles, and the proof of this analog is based on the “folding of the fibers” bound from Chapter 6.

We provide two applications of these finiteness theorems to nonnegative curvature. First, there are only finitely many vector bundles of a fixed rank over a fixed soul (that is, a fixed compact Riemannian manifold of nonnegative curvature) which are capable of admitting nonnegatively curved metrics for which the vertical curvatures at the soul satisfy a fixed upper bound. Second, most vector bundles over Bieberbach manifolds do not admit metrics of nonnegative curvature.

7.1 Introduction

Guijarro and Walschap recently proved that for fixed $n, k \in \mathbb{Z}$ and $\Lambda \in \mathbb{R}$, if M^{n+k} is an open manifold of nonnegative curvature whose soul is isometric to $S := S^n(1)$, and the curvature of M is bounded above by Λ , then there are only finitely many possibilities for the isomorphism class of the normal bundle of the soul in M [18, Theorem A]. The purpose of this chapter is 1) to prove this theorem for an arbitrary soul S , and 2) to use variations on Guijarro and Walschap’s idea to produce finiteness theorems for vector bundles and principal bundles with connections.

We begin in section 7.2 by proving the following finiteness theorem for vector bundles with connections:

Theorem 7.1.1. *For any set of positive constants $V, D, \lambda, \Lambda, n, k$ ($n, k \in \mathbb{Z}$), denote by $N = N(V, D, \lambda, \Lambda, n, k)$ the number of isomorphism classes of rank k vector bundles $\mathbb{R}^k \rightarrow E \rightarrow B^n$ which admit the following structure:*

1. *A metric on B for which $\text{vol}(B) \geq V$, $\text{diam}(B) \leq D$, $|\text{sec}(B)| \leq \lambda$.*
2. *A Euclidean structure (i.e., a smoothly varying inner product on the fibers).*
3. *A connection ∇ which is compatible with the Euclidean structure, whose associated curvature tensor, R^∇ , satisfies $|R^\nabla| \leq \Lambda$ (here $|R^\nabla|$ is defined with respect to the metric on B and the inner products on the fibers).*

Then $N \leq C \cdot (1 + \Lambda)^c$, where c, C depend only on V, D, λ, n, k .

Notice that Cheeger's Finiteness Theorem implies that there are only finitely many possibilities for the diffeomorphism type of the base space B , which is necessary for this theorem to make sense. Notice also that, for a fixed base B with a fixed metric, a bound on $|R^\nabla|$ implies bounds on the characteristic classes of the bundle, which implies that there are only finitely many possibilities for the isomorphism class of the bundle; see for example [1, Chapter 7]. However, it seems difficult to obtain from this line of reasoning an explicit upper bound for N in terms of Λ as in Theorem 7.1.1.

Using Corollary 3.1.8, we get as an immediate corollary to Theorem 7.1.1 the following version of Guijarro and Walschap's result for arbitrary souls:

Corollary 7.1.2. *Suppose that M^{n+k} is an open manifold of nonnegative sectional curvature such that:*

1. *The soul, Σ^n , of M satisfies: $\text{vol}(\Sigma) \geq V$, $\text{diam}(\Sigma) \leq D$, $\text{sec}(\Sigma) \leq \lambda$.*
2. *The curvature of every vertical 2-plane at every point of Σ is $\leq \Lambda$.*

Then the number, $N = N(V, D, \lambda, \Lambda, n, k)$, of possibilities for the isomorphism class of the normal bundle of the soul is $\leq C \cdot (1 + \Lambda)^c$, where c, C depend only on V, D, λ, n, k .

By a vertical 2-plane we mean a plane spanned by two vectors which are both normal to the soul.

Notice that Corollary 7.1.2 is stated somewhat carefully. It would be a stronger statement to say that "there are only finitely many isomorphism classes of vector bundles which admit a metric of nonnegative curvature with fixed bounds on the geometry of the soul and on the curvatures of vertical 2-planes at the soul". This statement would be stronger than Corollary 7.1.2 because the total space of a given vector bundle ξ could conceivably admit a metric of nonnegative curvature, but not one for which the soul is the zero section, and hence not one for which the normal bundle of the soul is necessarily isomorphic to ξ . The problem is that two vector bundles with diffeomorphic total spaces need not be isomorphic as vector bundles.

Next, in section 7.3, we prove the following analog of Theorem 7.1.1 for principal bundles:

Theorem 7.1.3. *For any compact Lie group G with a fixed bi-invariant metric, and any set of constants $V, D, \lambda, \Lambda, n$ ($n \in \mathbb{Z}$), let $N := N(G, V, D, \lambda, \Lambda, n)$ denote the number of isomorphism classes of principal G -bundles $G \rightarrow P \rightarrow B^n$ which admit the following structure:*

1. *A metric on B for which $\text{vol}(B) \geq V$, $\text{diam}(B) \leq D$, $|\text{sec}(B)| \leq \lambda$.*
2. *A connection, ω , whose curvature form, Ω , satisfies $|\Omega| \leq \Lambda$ (here $|\Omega|$ is defined with respect to the metrics on B and G).*

Then $N \leq e^{c(1+e^{C\Lambda})}$, where c, C depend only on G, V, D, λ, n .

Notice that Theorem 7.1.3 does not quite imply Theorem 7.1.1 because the bound is linear in the latter theorem, but double exponential in the former. We do not know whether the double exponential bound can be improved to exponential or linear.

Our proof of Theorem 7.1.3, viewed properly, provides a solution to the following problem: given a compact manifold B^n and a compact Lie group G , find a bound (depending on B and G) for the number of isomorphism classes of principal G -bundles over B which admit a flat connection. Notice that B and G do not have metrics in this problem, which is why the statement of Theorem 7.1.3 is not helpful. Nevertheless, the proof of Theorem 7.1.3 allows us to find a bound which depends on following invariants of B^n and G :

1. $I(B) :=$ the size of a minimal atlas for B (that is, the minimal number of open sets diffeomorphic to \mathbb{R}^n necessary to cover B).

2. $P(G, n) :=$ the number of points necessary to form an $\frac{\text{inj}(G)}{2^{n+4}-8}$ -net in (G, g_0) , where g_0 denotes the bi-invariant metric associated to the killing form on G .

Specifically, we prove the following:

Theorem 7.1.4. *Let B^n be a compact manifold and let G be compact Lie group. Then the number of isomorphism classes of principal G -bundles over B which are capable of admitting a flat connection is finite and is $\leq P(G, n)^{I(B)^2}$.*

As a consequence of Theorem 7.1.4, we prove that most vector bundles over Bieberbach manifolds (i.e., compact manifolds which are capable of admitting flat metrics) do not admit metrics of nonnegative curvature. More precisely,

Corollary 7.1.5. *Let S^n be a Bieberbach manifold, and let $k \in \mathbb{Z}$. Let $N = N(S, k)$ denote the number of isomorphism classes of rank k vector bundles over S which admit a metric of nonnegative curvature for which the zero section is the soul. Then N is finite and is in fact $\leq P(O(k), n)^{I(S)^2}$.*

This result is related to a theorem of Walschap and Özaydin, who proved that no nontrivial orientable rank 2 vector bundle over the torus T^n can admit a metric of nonnegative curvature [42]. As they showed, a vector bundle over a Bieberbach manifold admits a metric of nonnegative curvature for which the zero-section is the soul iff it admits a flat connection. To the best of our knowledge, it is reasonable to ask whether any nontrivial vector bundle over any Bieberbach manifold can admit a flat connection (and hence a metric of nonnegative curvature). Corollary 7.1.5 provides evidence that the answer might be no.

Finally, we address the question of whether Theorem 7.1.3 is valid when G is a noncompact Lie group with a left-invariant metric.

7.2 A finiteness theorem for vector bundles

In this section we prove Theorem 7.1.1. The proof depends on the following result of Gromov ([10, 2.16], [11, 0.5.E]):

Lemma 7.2.1 (Gromov).

1. (Easy version) *Let Y be a compact path-connected length space which satisfies the following regularity condition: there exists $\delta = \delta(Y)$ such that any two maps f_1, f_2 from the same space into Y with $d_{\text{sup}}(f_1, f_2) \leq \delta$ must be homotopic. Let X^n be a compact Riemannian manifold. Then the number, N , of homotopy classes of maps $X \rightarrow Y$ which admit an L -Lipschitz representative is $\leq e^{cL^n}$, where c depends on X and Y .*
2. (More difficult version) *If Y is a compact Riemannian manifold with a finite fundamental group, then $N \leq C \cdot (1+L)^c$, where the exponent c depends only on the homotopy types of X and Y , while the constant C depends on their metrics as well.*

Later, in the proof of Theorem 7.1.3, we will need to copy the idea of Gromov's proof of part 1 of Lemma 7.2.1; in anticipation of this, we include here the proof:

Proof of Part 1 of Lemma 7.2.1. Let $\delta := \delta(Y)$. Let R_Y denote a $\frac{\delta}{4}$ -net in Y , and let R_X denote a $\frac{\delta}{4L}$ -net in X . Any map $f : X \rightarrow Y$ induces a (non-unique) map $\hat{f} : R_X \rightarrow R_Y$, defined so that $\hat{f}(p)$ is any point of R_Y whose distance from $f(p)$ is $\leq \frac{\delta}{4}$. If $f_1, f_2 : X \rightarrow Y$ are two L -Lipschitz maps for which $\hat{f}_1 = \hat{f}_2$, it is easy to see that $d_{\text{sup}}(f_1, f_2) \leq \delta$, so f_1 is homotopic to f_2 . This

proves that the number, $N(X, Y, L)$, of distinct homotopy classes of maps $X \rightarrow Y$ which admit an L -Lipschitz representative is $\leq \text{card}(R_Y)^{\text{card}(R_X)}$.

Next we establish a bound for $\text{card}(R_X)$. Denote by $\text{cap}_r(X)$ the minimal cardinality of an r -net in X . We prove that $\text{cap}_r(X) \leq K((\frac{1}{r})^n + 1)$, where K depends on X . Notice that without the “+1” term, this bound would be incorrect when r is very large. Let B_n denote the open ball of radius 1 in \mathbb{R}^n . Choose an atlas for X ; that is, an open covering $\{U_i\}$ of X , and diffeomorphisms $\Phi_i : B_n \rightarrow U_i$, $i = 1, \dots, l$. Let \mathcal{L} denote a common Lipschitz bound for the maps Φ_i . Clearly,

$$\text{cap}_r(X) \leq \sum_{i=1}^l \text{cap}_r(U_i) \leq l \cdot \text{cap}_{(\frac{r}{\mathcal{L}})}(B_n) \stackrel{(r \rightarrow 0)}{\sim} l \cdot \mathcal{K} \cdot \left(\frac{\mathcal{L}}{r}\right)^n,$$

where \mathcal{K} depends only on n . In particular, we can choose R small enough so that for any $r < R$, $\text{cap}_r(X) \leq K'(\frac{1}{r})^n$, where $K' := 2 \cdot l \cdot \mathcal{K} \cdot \mathcal{L}^n$. So defining $K := \max\{K', \text{cap}_R(X)\}$ will insure that $\text{cap}_r(X) \leq K((\frac{1}{r})^n + 1)$ for any r .

Thus, $N(X, Y, L) \leq \text{card}(R_Y)^{K((\frac{4L}{\delta})^n + 1)} \leq e^{c(L^n + 1)}$ for properly chosen c . Since $N(X, Y, L) = 1$ for L sufficiently close to zero (specifically for $L < \frac{\delta}{\text{diam}(X)}$), it is easy to see that c can be redefined such that $N(X, Y, L) \leq e^{cL^n}$. \square

Proof of Theorem 7.1.1. Let B^n be a compact Riemannian manifold, and let $E \xrightarrow{\pi} B$ be a rank k vector bundle endowed with a Euclidean structure and a compatible connection, ∇ . Assume that $|R^\nabla| \leq \Lambda$. We prove that the number of candidates for the isomorphism class of this bundle is $\leq e^{(c+C\Lambda)^n}$, where c, C depend only on B and k . This result implies Theorem 7.1.1 by the following argument: given bounds V, D, λ, n , let $\mathfrak{M} = \mathfrak{M}(V, D, \lambda, n)$ denote the class of Riemannian n -manifolds satisfying $\text{vol} \geq V$, $\text{diam} \leq D$, and $|\text{sec}| \leq \lambda$. \mathfrak{M} is precompact in the Lipschitz topology and contains only finitely many diffeomorphism types (see [25]). This means that it is possible to choose a finite collection $\{M_1, \dots, M_m\} \subset \mathfrak{M}$ and a constant $L = L(V, D, \lambda, n)$ such that for any $M \in \mathfrak{M}$ there exists an L -biLipschitz diffeomorphism from M to some M_i . Therefore, any vector bundle over any manifold $M \in \mathfrak{M}$ which admits a Euclidean structure and a compatible connection satisfying the bound $|R^\nabla| \leq \Lambda$ is isomorphic to a vector bundle over some M_i which admits a Euclidean structure and a connection satisfying $|R^\nabla| \leq L^2 \cdot \Lambda$; namely, the pull-back bundle with the pull back Euclidean structure and connection. From this observation it is easy to see that the above stated “fixed base space” version of the theorem implies the more general version of the theorem.

We begin by choosing an explicit atlas of local trivializations for the bundle $E \xrightarrow{\pi} B$. Choose a value $r < \frac{1}{2} \min\{\text{inj}(B), \frac{\pi}{\sqrt{\lambda}}\}$, where λ denotes an upper curvature bound for B . Notice that any Jacobi field, $J(t)$, along any unit-speed geodesic of B with $J(0) = 0$ has monotonically increasing norm for $t \in [0, r]$; This follows, for example, from the version of Rauch’s theorem stated in [3, Lemma 6.3.5]. Choose a finite collection $\{p_1, \dots, p_l\}$ of points of B such that the collection of r -balls $\{B_i := B_{p_i}(r)\}$ covers B . Let $\{\eta_i\}$ denote a partition of unity subordinate to $\{B_i\}$.

The connection, ∇ , provides a natural collection of local trivializations, $\Phi_i : B_i \times \mathbb{R}^k \rightarrow \pi^{-1}(B_i)$. More precisely, for each i , choose a fixed orthogonal identification $s_i : \mathbb{R}^k \rightarrow E_{p_i} := \pi^{-1}(p_i)$, and define $\Phi_i(q, v)$ to be the parallel transport of $s_i(v)$ along the minimal geodesic, γ_q^i , between p_i and q . That is, $\Phi_i(q, v) := P_{\gamma_q^i}(s_i(v))$. Also define $h_i : \pi^{-1}(B_i) \rightarrow \mathbb{R}^k$ as $h_i := p_i \circ \Phi_i^{-1}$, where $p_i : B_i \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ denotes the projection onto the second factor. Notice that $h_i(v) = s_i^{-1}(P_{(\gamma_{\pi(v)}^i)^{-1}}(v))$.

For any pair (i, j) for which $B_{ij} := B_i \cap B_j$ is nonempty, we have a transition function $g_{ij} : B_{ij} \rightarrow O(k)$, defined as

$$g_{ij}(q)(v) := h_j(h_i^{-1}(v) \cap \pi^{-1}(q)) = (s_j^{-1} \circ P_{(\gamma_q^j)^{-1}} \circ P_{\gamma_q^i} \circ s_i)(v).$$

We wish to establish a Lipschitz bound on each g_{ij} . Suppose that $q \in B_{ij}$, and let $X \in T_q B$ with $|X| = 1$. Let $q(s) := \exp(sX)$ and let $v \in \mathbb{R}^k$ be any unit-length vector. Let $\tau_v(s) := g_{ij}(q(s))(v)$,

which is a path on the unit sphere of \mathbb{R}^k . In order to bound $|(g_{ij})_*(X)|$, it will clearly suffice to find a bound on $|\tau'_v(0)|$ which does not depend on the choice of v .

For small values of s , let $t \rightarrow \sigma_s(t) = \sigma(s, t)$ be a minimal unit-speed path between p_i and $q(s)$ followed by a minimal unit-speed path between $q(s)$ and p_j . By construction, $\tau_v(s) = (s_j^{-1} \circ P_{\sigma_s} \circ s_i)(v)$. Let $\text{area}(s_0)$ denote the area of the homotopy between σ_0 and σ_{s_0} determined by the family of curves σ_s , $s \in [0, s_0]$. By [3, Proposition 6.2.1], $\angle(\tau_v(0), \tau_v(s_0)) \leq \Lambda \cdot \text{area}(s_0)$. Therefore $|\tau'_v(0)| \leq \Lambda \cdot \text{area}'(0)$. But $\text{area}'(0) = \int_{t=0}^{\text{dist}(p_i, p_j)} |\frac{\partial}{\partial s} \sigma(0, t)| ds \leq \text{dist}(p_i, p_j) \leq 2r$. To see the first inequality, notice that the vector field $\frac{\partial}{\partial s} \sigma$ is a Jacobi field along γ_q^i and along γ_q^j , and that both Jacobi fields have norms which vary monotonically between 0 and $|X| = 1$. Therefore, $|\tau'_v(0)| \leq 2r\Lambda$, which implies that $|(g_{ij})_*(X)| \leq 2kr\Lambda$. In other words, each transition function g_{ij} is $(2kr\Lambda)$ -Lipschitz.

The isomorphism class of the vector bundle $E \xrightarrow{\pi} B$ is determined by the homotopy type of its classifying map $f : B \rightarrow G_k(\mathbb{R}^N)$, where $G_k(\mathbb{R}^N)$ denotes the Grassmannian of k -dimensional subspaces of \mathbb{R}^N , and $N = k \cdot l$. The map f can be described explicitly in terms of our chosen atlas of trivializations of the bundle as follows (see for example [19, Theorem 3.5.5]): think of $\mathbb{R}^N = \sum_1^l \mathbb{R}^k$, and define

$$f(p) := \{(\eta_1(p) \cdot h_1(v), \dots, \eta_l(p) \cdot h_l(v)) \mid v \in \pi^{-1}(p)\}.$$

Our goal is to establish a Lipschitz bound for f which is linear in Λ . For this purpose, the most useful description of the homogeneous metric on $G_k(\mathbb{R}^N)$ is as follows. Fix orthonormal bases of \mathbb{R}^N and \mathbb{R}^k . Let I_k denote the $k \times k$ identity matrix, and let $I_k^N := \begin{pmatrix} I_k \\ 0 \end{pmatrix}$ denote the $N \times k$ matrix whose top k rows form I_k , and whose bottom $N - k$ rows are zeros. An arbitrary point, V , of $G_k(\mathbb{R}^N)$ can be described as $V = \text{im}(P \cdot I_k^N)$ for some $P \in O(N)$. An arbitrary path, $\gamma(t)$, with $\gamma(0) = V$ can be described as $\gamma(t) = \text{im}(P \cdot A(t))$, where $A(t) : \mathbb{R}^k \rightarrow \mathbb{R}^N$ is a family of linear embeddings with $A(0) = I_k^N$. Then $|\gamma'(0)|^2 = \text{trace}(B^T \cdot B)$, where $B := A'(0) - I_k^N \cdot (I_k^N)^T \cdot A'(0)$ (that is, B is obtained from the matrix $A'(0)$ by replacing the top k rows with zeros). In particular, if all components $a_{ij}(t)$ of $A(t)$ satisfy $|a'_{ij}(0)| \leq c$, then $|\gamma'(0)| \leq kc$.

Now fix $q \in B_1$ and $X \in T_q B$ with $|X| = 1$, and let $q(t) = \exp(tX)$. Let $A(t) = (a_{ij}(t)) : \mathbb{R}^k \rightarrow \mathbb{R}^N$ be defined as

$$A(t)(v) := (\eta_1(q(t)) \cdot v, \eta_2(q(t)) \cdot g_{12}(q(t))(v), \dots, \eta_l(q(t)) \cdot g_{1l}(q(t))(v)).$$

It is clear that $f(q(t)) = \text{im}(A(t))$. Choose $S = (s_{ij}) : \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that the matrix $A(0) \cdot S$ is an orthogonal map onto its image. That way, $P \cdot A(0) \cdot S = I_k^N$ for some $P = (p_{ij}) \in O(N)$. Since $A(0)$ already maps an orthonormal basis of \mathbb{R}^k onto an orthogonal set in \mathbb{R}^N , S can be chosen as a diagonal matrix, and it is clear that the components of S satisfy $|s_{ij}| \leq k$. $f(q(t)) = \text{im}(A(t)) = \text{im}(P^{-1}P \cdot A(t) \cdot S)$. So, by the previous discussion of the homogeneous metric on $G_k(\mathbb{R}^N)$:

$$\begin{aligned} |f_*(X)| &\leq k \cdot \max |((P \cdot A \cdot S)'(0))_{ij}| \\ &\leq k^2 N (\max |p_{ij}|) \cdot (\max |a'_{ij}(0)|) \cdot (\max |s_{ij}|) \\ &= k^3 N (\max |a'_{ij}(0)|) \\ &\leq k^3 N [(\max |(\eta_i)_*(X)|) \cdot (\max |(g_{1s})_{ij}(q)|) \\ &\quad + (\max |(\eta_i(q))|) \cdot (\max |((g_{1s})_{ij})_*(X)|)] \\ &\leq k^3 N (\hat{L} \cdot 1 + 1 \cdot 2r\Lambda) = k^3 N (\hat{L} + 2r\Lambda), \end{aligned}$$

where \hat{L} denotes a Lipschitz bound on the functions η_i .

Thus, f satisfies the Lipschitz constant $L := k^3 N (\hat{L} + 2r\Lambda)$. The homotopy type of this map classifies the bundle, so the proof is completed by part 2 of Lemma 7.2.1. \square

Example 7.2.2. There is a countable collection of rank 2 vector bundles over S^2 , which can be described as $M_k := (S^3 \times \mathbb{R}^2)/S^1$, $k \in \mathbb{Z}^+$, where S^1 acts diagonally as

$$(p, v) \mapsto (e^{i\theta} \cdot p, e^{-ki\theta} \cdot v).$$

Notice that M_1 is isomorphic to the normal bundle of \mathbb{CP}^1 in \mathbb{CP}^2 , and M_2 is isomorphic to TS^2 . The quotient metric on M_k has nonnegative curvature by O'Neill's formula. Denote by $[p, v]$ the equivalence class in M_k of $(p, v) \in S^3 \times \mathbb{R}^2$. It is easy to see that the soul Σ of M_k is $\{[p, 0] \mid p \in S^3\}$, which is isometric to $S^3/S^1 = S^2(\frac{1}{2})$. The metric projection $\pi : M_k \rightarrow \Sigma$ acts as $\pi([p, v]) = [p, 0]$. Let $q = [p, v] \in M_k$. The distance, r , between q and Σ in M_k is just $r = |v|$. By studying the double Riemannian submersion $S^3 \times \mathbb{R}^2 \rightarrow M_k \rightarrow \Sigma$, it is straightforward to compute that the norm of the A tensor of π at q is $|A_q| = \frac{kr}{\sqrt{1+k^2r^2}} =: f_k(r)$. Therefore, for any $p \in \Sigma$, $V \in \nu_p(\Sigma)$ and $X, Y \in T_p\Sigma$ with $|X|, |Y|, |V| = 1$ and $X \perp Y$, $|R^\nabla(X, Y)V| = 2|(D_V A)_X Y| = 2f'_k(0) = 2k$. In other words, $|R^\nabla| = 2k$. Each M_k has totally geodesic π -fibers. By Corollary 7.1.2, the vertical curvatures at their souls must also grow proportionally to k , as can be verified computationally.

7.3 A finiteness theorem for principal bundles

In this section we prove Theorem 7.1.3. Our proof begins with a lemma which essentially says that two principal G -bundles whose transition functions are sufficiently close must be isomorphic.

Lemma 7.3.1. *Let G be a compact Lie group with a bi-invariant metric. Let $\delta := \frac{1}{2}\text{inj}(G)$. Let B be a compact manifold. Let $\{B_1, \dots, B_l\}$ be a covering of B by contractible open sets. Let z denote the maximal multiplicity of intersections (that is, no point of B lies in more than z of the open sets). Let $\pi^0 : P^0 \rightarrow B, \pi^1 : P^1 \rightarrow B$ be two principal G -bundles over B . Let $g_{ij}^0, g_{ij}^1 : B_i \cap B_j \rightarrow G$ denote the transition functions associated to atlases of trivializations of the two bundles over the sets B_i . If for all pairs (i, j) and for all $p \in B_i \cap B_j$, $d_G(g_{ij}^0(p), g_{ij}^1(p)) \leq \epsilon := \frac{\delta}{2^z - 1}$, then the two bundles are isomorphic.*

Our proof will use only the following properties of (G, d_G, δ) :

1. $d_G((a')^{-1} \cdot b', a^{-1} \cdot b) \leq d_G(a', a) + d_G(b', b)$ for all $a, a', b, b' \in G$.
2. For any two points $a, b \in G$ with $d_G(a, b) \leq \delta$, there is a canonical path $\gamma_{ab} : [0, 1] \rightarrow G$ from a to b (namely the minimal path) which varies continuously with a and b and for which $d_G(\gamma_{ab}(t), a), d_G(\gamma_{ab}(t), b) \leq d_G(a, b)$.

Proof. We begin by reviewing Milnor's construction of the classifying space for a principal G -bundle (see [19, Section 4.11]). Define:

$$E_G(l) := \{(t_1 g_1, \dots, t_l g_l) \mid g_i \in G, t_i \in [0, 1], \sum t_i = 1\} / \sim,$$

where two vectors, $(t_1 g_1, \dots, t_l g_l)$ and $(\tilde{t}_1 \tilde{g}_1, \dots, \tilde{t}_l \tilde{g}_l)$, are considered equivalent iff for each i , either 1) $g_i = \tilde{g}_i$ and $t_i = \tilde{t}_i$ or 2) $t_i = \tilde{t}_i = 0$. Then define $B_G(l) := E_G(l)/G$, where G acts on $E_G(l)$ component-wise on the left. Suppose $\pi : P \rightarrow B$ is a principal G -bundle over B , and $\Phi_i : B_i \times G \rightarrow \pi^{-1}(B_i)$ is an atlas of trivializations for π over the open covering $\{B_1, \dots, B_l\}$ of B . Let $\{\eta_1, \dots, \eta_l\}$ denote a partition of unity subordinate to the open cover $\{B_1, \dots, B_l\}$. Define $h_i : \pi^{-1}(B_i) \rightarrow G$ as $h_i := p_i \circ \Phi_i^{-1}$, where $p_i : B_i \times G \rightarrow G$ denotes the projection onto the second factor. Let $g_{ij} : B_i \cap B_j \rightarrow G$ denote the transition functions associated to this collection of trivializations.

Define $\tilde{f} : P \rightarrow E_G(l)$ as follows:

$$\tilde{f}(z) := (\eta_1(\pi(z)) \cdot h_1(z), \dots, \eta_l(\pi(z)) \cdot h_l(z)).$$

\tilde{f} maps fibers to fibers, so it induces a well-defined map $f : B \rightarrow B_G(l)$. f is called the classifying map of the bundle. It is useful to describe f in terms only of the transition functions. For example, f restricted to B_1 can be described as follows:

$$f(p) = [\eta_1(p) \cdot e, \eta_2(p) \cdot g_{12}(p), \dots, \eta_l(p) \cdot g_{1l}(p)],$$

where e denotes the identity element of G . Of course, f satisfies the analogous description when restricted to the other balls B_2, \dots, B_l .

Now let $\pi^0 : P^0 \rightarrow B, \pi^1 : P^1 \rightarrow B$ be two principal G -bundles over B . Let $g_{ij}^0, g_{ij}^1 : B_i \cap B_j \rightarrow G$ denote the transition functions associated to atlases of trivializations of the two bundles over the same atlas $\{B_1, \dots, B_l\}$ of B . Let $f^0, f^1 : B \rightarrow B_G(l)$ denote their classifying maps, constructed as above using the same partition of unity $\{\eta_i\}$. Assume that for all pairs (i, j) and for all $p \in B_i \cap B_j$, $d_G(g_{ij}^0(p), g_{ij}^1(p)) \leq \epsilon$. In order to define a homotopy, f^t , between f^0 and f^1 , first define for every pair (i, j) and for every $q \in B_i \cap B_j$, $g_{ij}^t(q)$ ($t \in [0, 1]$) to be the canonical path in G between $g_{ij}^0(q)$ and $g_{ij}^1(q)$. Next define f^t restricted to B_1 as:

$$f^t(p) = [\eta_1(p) \cdot e, \eta_2(p) \cdot g_{12}^t(p), \dots, \eta_l(p) \cdot g_{1l}^t(p)]. \quad (7.3.1)$$

It is tempting to define f^t restricted to the other balls B_2, \dots, B_l analogously; however, this definition would only be well defined on the overlaps when $t = 0$ and $t = 1$. So instead we must extend f^t one open set at a time. For example, extend f^t to $B_1 \cup B_2$ as follows. First notice that for $p \in B_1 \cap B_2$, the following description of f^t is equivalent to equation 7.3.7.1:

$$f^t(p) = [\eta_1(p) \cdot (g_{12}^t(p))^{-1}, \eta_2(p) \cdot e, \dots, \eta_l(p) \cdot (g_{1l}^t(p))^{-1} \cdot g_{1l}^t(p)]. \quad (7.3.2)$$

The formula has just been rewritten so that e appears in the second component, which makes it easier to compare to the natural definition of f^t restricted to B_2 :

$$f^t(p) = [\eta_1(p) \cdot g_{21}^t(p), \eta_2(p) \cdot e, \dots, \eta_l(p) \cdot g_{2l}^t(p)]. \quad (7.3.3)$$

At time $t = 0$ and $t = 1$, equations 7.3.7.2 and 7.3.7.3 agree on $B_1 \cap B_2$. For $t \in (0, 1)$, we redefine f^t on $B_1 \cap B_2$ as the weighted average of equations 7.3.7.2 and 7.3.7.3. More precisely, for each of the l components in these two expressions, the coefficients are equal and the elements of G have distance $\leq 3\epsilon \leq \delta$ in G . To see this, consider, for example, the group elements in the l^{th} component:

$$\begin{aligned} & d_G((g_{12}^t(p))^{-1} \cdot g_{1l}^t(p), g_{2l}^t(p)) \\ & \leq d_G((g_{12}^t(p))^{-1} \cdot g_{1l}^0(p), g_{2l}^0(p)) + d_G(g_{2l}^0(p), g_{2l}^t(p)) \\ & \leq d_G((g_{12}^0(p), g_{1l}^0(p)) + d_G(g_{1l}^0(p), g_{1l}^t(p)) + d_G(g_{2l}^0(p), g_{2l}^t(p)) \\ & \leq \epsilon + \epsilon + \epsilon = 3\epsilon \leq \delta. \end{aligned} \quad (7.3.4)$$

So it makes sense to take a weighted average of the two elements of G , using $\frac{\eta_1}{\eta_1 + \eta_2}$ and $\frac{\eta_2}{\eta_1 + \eta_2}$ as the weights (by the weighted average of two nearby points $a, b \in G$ we mean the point $\gamma_{ab}(t_0)$, where t_0 is determined in the obvious way by the weights). To make the weights well defined, choose a partition of unity such that $\eta_i(p) > 0$ for all p in the interior of B_i .

Finally, extend the definition of f^t to the other balls B_3, \dots, B_l one at a time in exactly the same way. That is, if f^t is already defined on $U = B_1 \cup \dots \cup B_{m-1}$, then describe a component-wise formula for f^t on $U \cap B_m$ with e in the m^{th} position. Then re-define f^t on $U \cap B_m$ as the component-wise weighted average of this formula and the natural definition for f^t on B_m . Use $\frac{\eta_1 + \dots + \eta_{m-1}}{\eta_1 + \dots + \eta_m}$ and $\frac{\eta_m}{\eta_1 + \dots + \eta_m}$ as the weights. The pairs of elements of G to be averaged will always agree when $t = 0$ and $t = 1$; further, by repeating the argument of equation 7.3.7.4 inductively, it is easy to see that they will have distance $\leq (2^z - 1)\epsilon = \delta$ in G for $t \in (0, 1)$. The homotopy f^t is clearly well-defined and continuous. Since the classifying maps of the two bundles are homotopic, the bundles are isomorphic. This proves the lemma. \square

Proof of Theorem 7.1.3. Let B^n be a compact Riemannian manifold, G a compact Lie group with bi-invariant metric, and $\pi : P \rightarrow B$ a principal G -bundle with a connection, ω , whose associated curvature form, Ω , satisfies $|\Omega| \leq \Lambda$. We prove that the number of candidates for the isomorphism class of this bundle is $\leq e^{c(1+e^{C\Lambda})}$, where c, C depend only on B and G . By the argument which began our proof of Theorem 7.1.1, this “fixed base space” version implies the more general version of Theorem 7.1.3.

There is a unique Riemannian metric on P for which π is a Riemannian submersion whose horizontal distribution is defined by ω , and each fiber $G_p := \pi^{-1}(p)$ is totally geodesic and isometric to G (with the given bi-invariant metric). Since the fibers are totally geodesic, $T = 0$. Further, the A tensor of π agrees with the curvature form, Ω , so $|A| \leq \Lambda$.

As before, choose $r < \frac{1}{2} \min\{\text{inj}(B), \frac{\pi}{\sqrt{\Lambda}}\}$, where λ denotes an upper curvature bound for B , and choose points $\{p_1, \dots, p_l\} \subset B$ such that the balls $\{B_i := B_{p_i}(r)\}$ cover B . Choose isometries $s_i : G \rightarrow G_{p_i}$. The natural trivializations, $\Phi_i : B_i \times G \rightarrow \pi^{-1}(B_i)$, are defined as $\Phi_i(q, v) := h^{\gamma_q^i}(s_i(v))$. Remember that γ_q^i denotes the minimal path from p_i to q , and $h^{\gamma_q^i} : G_{p_i} \rightarrow G_q$ denotes the diffeomorphism between the fibers which is naturally associated to this path. Let $g_{ij} : B_{ij} = B_i \cap B_j \rightarrow G$ denote the transition functions associated to this atlas of trivializations of the bundle.

We require a new argument to establish Lipschitz control on the transition functions. As before, let $q \in B_{ij}$, $X \in T_q B$ with $|X| = 1$, and $q(s) = \exp(sX)$. As before, let $t \rightarrow \sigma_s(t) = \sigma(s, t)$ be a minimal unit-speed path between p_i and $q(s)$ followed by a minimal unit-speed path between $q(s)$ and p_j . Let $\tau(s) := g_{ij}(q(s))$. Notice that $\tau(s) = (s_j^{-1} \circ h^{\sigma_s} \circ s_i)(e)$, where e denotes the identity element of G . Denote $k := \dim(G)$. Since $C_1 := \text{length}(\sigma_0) \leq 2r$, and since $C_2 := \sup_{t \in [0, d(p_i, p_j)]} |\frac{\partial}{\partial s} \sigma(0, t)| \leq 1$ as before, Lemma 6.1.3 and Equation 6.1.6.2 provides the following bound:

$$|\tau'(0)| = |(s_j \circ \tau)'(0)| \leq 2e^{2rk4^k(k+1)!\Lambda} =: L.$$

This value L provides a Lipschitz bound for the transition functions, g_{ij} .

The most obvious strategy for completing the proof is the following: 1) describe a natural metric on the classifying space $B_G(l)$, 2) establish a Lipschitz bound for the classifying map $f : B \rightarrow B_G(l)$ depending on Lipschitz bounds for the transition functions and for the partition of unity functions, 3) verify that $B_G(l)$ satisfies the regularity hypothesis of part 1 of Lemma 7.2.1, and apply this lemma. We believe that this approach works. But in the case where $\Lambda = 0$ (the flat bundle case), the bound obtained from this approach for the number of isomorphism class possibilities of the bundle would depend on the metrics of B and G . Therefore, we use a different approach in order to obtain bounds in the flat bundle case which depend only on the topology of B and the Lie group structure of G . Rather than applying Gromov’s lemma, we must use a variation on its proof.

The variation of Gromov’s argument goes as follows. Let $\delta := \frac{1}{2} \text{inj}(G)$. Let X denote the disjoint union of all of the nonempty intersections B_{ij} . Let N denote the number of components of X , and let z denote the maximal multiplicity of intersections for the open cover $\{B_i\}$ of B . Let $\epsilon := \frac{\delta}{2z-1}$. Let R_G denote a $\frac{\epsilon}{4}$ -net in G , and let R_X denote a $\frac{\epsilon}{4L}$ -net in X (remember $L = 2e^{2rk4^k(k+1)!\Lambda}$ is a Lipschitz bound on the transition functions). As argued in our proof of Lemma 7.2.1, the net R_X can be chosen such that $\text{card}(R_X) \leq K \cdot \left(\left(\frac{4L}{\epsilon} \right)^n + 1 \right)$, where $K = K(X)$. Think of the collection $\{g_{ij}\}$ of transition functions as defining a single L -Lipschitz map $g : X \rightarrow G$. g induces a (non-unique) map $\hat{g} : R_X \rightarrow R_G$, defined so that $\hat{g}(p)$ is any point of R_G whose distance from $g(p)$ is $\leq \frac{\epsilon}{4}$. It is easy to see that if $g^0, g^1 : X \rightarrow G$ are any two L -Lipschitz maps, and $\hat{g}^0 = \hat{g}^1$, then $d_{\text{sup}}(g^0, g^1) \leq \epsilon$. If g^0 and g^1 happen to be the transition functions associated to two different bundles, $\pi^0 : P^0 \rightarrow B$ and $\pi^1 : P^1 \rightarrow B$, then Lemma 7.3.1 implies that the two bundles must be isomorphic. Thus, the number of candidates for the isomorphism class of our bundle is $\leq \text{card}(R_G)^{\text{card}(R_X)} \leq \text{card}(R_G)^{K((\frac{4L}{\epsilon})^n + 1)}$. Substituting the above value of L and simplifying completes the proof. \square

To conclude this section, discuss the case $\Lambda = 0$ in the previous proof in order to prove Theorem 7.1.4

Proof of Theorem 7.1.4. When $\Lambda = 0$, the transition functions satisfy the Lipschitz bound $L = 0$, so we can use a net R_X in X containing exactly one point from each component of X . In this case, there is no reason to use a coverings of B by small metric balls; any covering by open sets diffeomorphic to \mathbb{R}^n would suffice. For example, if we use a minimal such covering, then $\text{card}(R_X) \leq I(B)^2$. It is easy to choose a minimal open covering for which the maximum multiplicity of intersections is $z = n + 1$. For convenience, we can work with the bi-invariant metric on G associated to the Killing form. R_G is defined in the above proof as an r -net in G , where $r = \epsilon/4 = \left(\frac{\frac{1}{2}\text{inj}(G)}{2^z - 1}\right) / 4 = \frac{\text{inj}(G)}{2^{n+4} - 8}$. By definition, $\text{card}(R_G) = P(G, n)$. This establishes Theorem 7.1.4. \square

We end this section by proving Corollary 7.1.5 as a consequence of Theorem 7.1.4.

Proof of Corollary 7.1.5. Suppose that M is an open manifold of nonnegative curvature whose soul, Σ , is homeomorphic to a Bieberbach manifold. By [6, Corollary 9.5], any metric of nonnegative curvature on a Bieberbach manifold is flat, so Σ is flat. It then follows from O'Neill's formula (2.4.2.1) that the A -tensor of the metric projection $\pi : M \rightarrow \Sigma$ vanishes, which implies by Lemma 3.1.1 that the connection on the normal bundle of Σ in M is flat. This flat connection induces a flat connection in the principal $O(k)$ -bundle associated to the normal bundle of the soul. Theorem 7.1.4 bounds the number of possibilities for the isomorphism type of the principal bundle, and hence also of the normal bundle of the soul. \square

We end this section by discussing whether Theorem 7.1.3 is still true when G is only a non-compact Lie groups with a left-invariant metric. In this case, we can still prove a finiteness theorem as long as we add the assumption that the base space is simply connected:

Theorem 7.3.2. *Let G be a non-compact Lie group with a left-invariant metric, B a compact simply connected Riemannian manifold, and Λ a constant. Then there exist only finitely many isomorphism classes of principal G -bundles $G \rightarrow P \rightarrow B$ which admit a connection, ω , whose curvature form, Ω , satisfies $|\Omega| \leq \Lambda$.*

Proof. The compactness assumption for G was used in the proof of Theorem 7.1.3 to insure that there exists a finite $\frac{\epsilon}{4}$ -net R_G in G . When G is non-compact, we use the assumption that B is simply connected to insure that the images of the transition functions all lie in a fixed compact subset G' of G ; thus, we can complete the proof by settling for a finite $\frac{\epsilon}{4}$ -net in G' rather than in G .

More precisely, with the setup from the proof of Theorem 7.1.3, we show that the functions s_i can be contrived so that $\text{im}(g)$ lies in the ball of radius R about e in G , where $g : X \rightarrow G$ denotes the collection of transition functions, and R depends only on B, Λ, k . The construction is as follows: choose s_1 arbitrarily, and for each $i = 2, \dots, l$, define $s_i = h^{\sigma_i} \circ s_1$, where σ_i is any minimal path in B from p_1 to p_i . This construction insures that for any pair (i, j) , any $q \in B_{ij}$, $s_1(g_{ij}(q)) = h^\alpha(s_1(e))$, where α denotes the path $\alpha := \sigma_i * \gamma_q^i * (\gamma_q^j)^{-1} * (\sigma_j)^{-1}$. Therefore,

$$\begin{aligned} d_G(e, g_{ij}(q)) &= d_{G_{p_1}}(s_1(e), s_1(g_{ij}(q))) \\ &\leq C_1 \cdot d_P(s_1(e), s_1(g_{ij}(q))) \\ &\leq C_1 \cdot \text{length}(\alpha) \\ &\leq C_1(2r + 2 \cdot \text{diam}(B)) =: R, \end{aligned}$$

Where $C_1 = C_1(B, \Lambda, k)$ is the constant in Lemma 6.1.1. It follows that $\text{im}(g) \subset G' := B_e(R)$.

A second difficulty is that Lemma 7.3.1 is not necessarily valid when G is a noncompact Lie group with a left-invariant metric. The problem is that, even though any two points of G whose distance is $\leq \delta = \frac{1}{2}\text{inj}(G)$ do have a canonical path between them, it is not necessarily true that:

$$d_G((a')^{-1} \cdot b', a^{-1} \cdot b) \leq d_G(a', a) + d_G(b', b) \text{ for all } a, a', b, b' \in G.$$

However, the following is true: There exists $C := C(R)$ such that:

$$d_G((a')^{-1} \cdot b', a^{-1} \cdot b) \leq C(d_G(a', a) + d_G(b', b)) \text{ for all } a, a', b, b' \in G'.$$

To see this inequality, let $\gamma_1(t)$ be a minimal path from a to a' and $\gamma_2(t)$ a minimal path from b to b' , $t \in [0, 1]$. Let $\gamma := \gamma_1^{-1} \cdot \gamma_2$, which is a path from $a^{-1} \cdot b$ to $(a')^{-1} \cdot b'$. By the product rule, $|\gamma'(t)| \leq |\gamma_2'(t)| + C|\gamma_1'(t)|$, where $C := |dR_{\gamma_1(t)^{-1}}| \cdot |dR_{\gamma_2(t)}|$. This weaker statement suffices to complete the proof. \square

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