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MATH 140 : CALCULUS II

Problem Set 6 – due Wednesday, February 26th

INSTRUCTIONS:
Please submit this at the *start* of Wednesday's class. Don't worry if you don't manage the get an answer for any particular question, but please give each problem an honest try (and record what you were able to accomplish, even if you didn't solve it). Eventually you should make sure to understand the problems, as some of them may appear on next week's in-class quiz. You are encouraged to collaborate with other students on these problems. However, please write up your solutions in isolation from one another.

6.1 Recall that $\sin^{-1} x$ is (by definition) the angle between $-\pi/2$ and $\pi/2$ whose sine equals x. In particular, for any x between -1 and 1 we have

$$\sin(\sin^{-1} x) = x.$$

(a) Show that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$. [*Hint: Draw a right triangle with hypotenuse 1 and one of its sides of length x.*]

(b) Determine (with justification!) $\frac{d}{dx} \sin^{-1} x$. [*The answer is $\frac{1}{\sqrt{1-x^2}}$.*]

6.2 Evaluate $\log_2 32$ without a calculator.

Problem 6.3 (below) is not going to appear on Monday's quiz. Instead, it is a bonus question. If you submit a solution before the first midterm, I will replace your lowest quiz score with your score on this problem (each part of the problem is worth one point; the whole problem is out of 8 points). Although you are welcome to brainstorm with other students, you must write up your solution in isolation from anyone else.

6.3 The goal of this exercise is to prove that $\frac{d}{dx} \sin x = \cos x$.

(a) Show that

$$\frac{d}{dx} \sin x = (\sin x) \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + (\cos x) \left(\lim_{h \to 0} \frac{\sin h}{h} \right).$$

[*We did this in class; I'm just asking you to explain what we did.*]

In the next few parts you will show that $\lim_{h \to 0} \frac{\sin h}{h} = 1$ using the following picture:

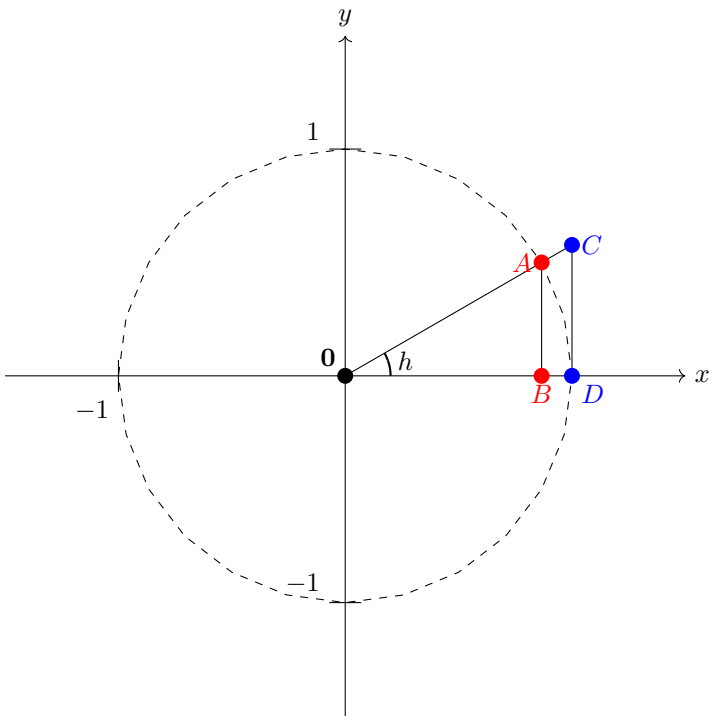

*Illustration for Problem **6.3**.*

(b) Find the area of triangle $A\mathbf{0}B$.

(c) Find the area of triangle $C\mathbf{0}D$. [*Hint: First show that the length of* $\overline{CD}$ *is* $\tan h$.]

(d) Find the area of sector (i.e. slice of the circle) $A\mathbf{0}D$. [*Hint: What proportion of the circle is it?*]

(e) Use the above to deduce that $\cos h \leq \frac{\sin h}{h} \leq \frac{1}{\cos h}$ [*Hint: First figure out how the three areas above compare to one another. Then: algebra time!*]

(f) Use the above to explain why $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$.

(g) Use previous part to show that $\lim_{h\to 0} \dfrac{\cos h - 1}{h} = 0$. [*Hint: Multiply by* $\frac{\cos h+1}{\cos h+1}$.]

(h) Use all of the previous work to conclude that $\frac{d}{dx} \sin x = \cos x$!