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MATH 140 : CALCULUS II

Problem Set 11 – due Monday, March 9th

INSTRUCTIONS:
Please submit this at the *start* of Monday's class. Don't worry if you don't manage the get an answer for any particular question, but please give each problem an honest try (and record what you were able to accomplish, even if you didn't solve it). Eventually you should make sure to understand the problems, as some of them may appear on next week's in-class quiz. You are encouraged to collaborate with other students on these problems. However, please write up your solutions in isolation from one another.

11.1 In class we tried to evaluate $\int_0^2 (4-x^2)\,dx$ using the limit definition, but got stuck at evaluating a certain sum. Complete our calculation by using the formula

$$1^2 + 2^2 + 3^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}.$$

In other words, write out the computation of the integral from start to finish; the first half of your solution should look like what we did in class (try to do it without looking at the notes!), and the second half should use the formula above. [*Ans:* $8 - \frac{8}{3}$]

11.2 In class we proved

$$1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{2}.$$

In this problem we explore a couple of other sums.

(a) Let $S = 1 + x + x^2 + x^3 + \cdots + x^n$. What's the simplest expression you can find for $xS - S$? Use this to obtain a nice formula for S.

(b) Differentiate both sides to obtain a formula for $1 + 2x + 3x^2 + \cdots + nx^{n-1}$. What happens when you plug in $x = 1$?

(c) Multiply both sides of your above formula by x and differentiate again to obtain a formula for $1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}$.

(d) Use the previous part to find a formula for $1^2 + 2^2 + 3^2 + \cdots + n^2$.

(e) Figure out a formula for $1^3 + 2^3 + 3^3 + \cdots + n^3$ using the above strategy.

11.3 In this problem we find the formula for the sum

$$1^2 + 2^2 + 3^2 + \cdots + n^2$$

using a different method from the previous problem, one which is close in spirit to our approach in class to finding $1 + 2 + 3 + \cdots + n$.

(a) Consider the triangle of numbers

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 3 & & 3 & & 3 \end{array}$$

Rotate this triangle counterclockwise by 120° (so that the 1 ends up in the lower left vertex and one of the 3's ends up in the top vertex), and also by 240°. What happens when you add together these three triangles? (Your answer should be a triangle of numbers.)

(b) Repeat the previous part with the triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & 2 & & & \\ & & 3 & & 3 & & 3 & & \\ & 4 & & 4 & & 4 & & 4 & \\ 5 & & 5 & & 5 & & 5 & & 5 \end{array}$$

(c) What would you get if you did the same with a triangle as above, but with n rows?

(d) What's the sum of all the numbers in the triangle with n rows?

(e) Derive a formula for $1^2 + 2^2 + 3^2 + \cdots + n^2$ from the above.