Number Theory I

Monday, February 4

Modern number theory comes from Gauss (Gauß). Plenty of ~~m~~ notation and proofs come from him. (He was about our age in 1800.) Once upon a time, Gauss was in school, and his teacher wanted to occupy him, which leads us to:

Evaluate: $1+2+3+\dots+99+100=?$

We can use arithmetic sequences, but how? Note that:

$\underbrace{1+\underbrace{2+\dots+99}_{101}+100}_{101}$ There are 50 such pairs, so this yields $101\cdot 50=5050$, which is what Gauss got.

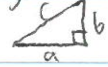
In general, what is $1+2+3+\dots+(n-2)+(n-1)+n$? 1 and n are the same distance from the middle term, but what if there is no middle term?

Let
$$\begin{aligned} S&=1+2+3+\dots+(n-2)+(n-1)+n\\ +\ S&=n+(n-1)+(n-2)+\dots+3+2+1\\ \hline 2S&=(n+1)+(n+1)+(n+1)+\dots+(n+1)+(n+1)+(n+1). \end{aligned}$$

How many $(n+1)$s are there? Since there are n terms in S, there are n $(n+1)$ terms, so $2S=n(n+1)$ and $S=\frac{n(n+1)}{2}$. This proves:

Theorem: $1+2+3+\dots+(n-1)+n=\frac{n(n+1)}{2}$.

Now onto something harder. We're going back in time! Pythagoras didn't prove anything, but he started the concept of math as a religion. The first proofs, so far as we know, come from the Pythagoreans. (Fun side note: there are plenty of crazy stories about Pythagoras. He allegedly bit a snake that just bit him and killed it, and he supposedly won an Olympic boxing event by being clever.)

Pythagorean Theorem: 

$a^2+b^2=c^2$. This leads us to a problem.

Problem:  How long is the diagonal? It's $\sqrt{2}$... but what is $\sqrt{2}$? (Note: $\sqrt{2}$ is interesting notation. It doesn't tell us anything about the number, but rather what it does.)

Is $\sqrt{2}$ a fraction? Guesses: $\left(\frac{3}{2}\right)^2 = \frac{9}{4} \neq 2$, $\left(\frac{7}{5}\right)^2 = \frac{49}{25} \neq 2$, $\left(\frac{17}{12}\right)^2 = \frac{289}{144} \neq 2$.

Hippasus (Ιππασος): Maybe $\sqrt{2}$ isn't a fraction! Maybe we can't write it in the form $\frac{a}{b}$ where a and b are integers! (Hippasus was drowned for this suggestion.)

Hippasus's suggestion was later proved in Euclid's <u>Elements</u>. We're going to prove it too! Before we do so, though, we'll do some scratch work.

What if we could find $\frac{a}{b}$ such that $\left(\frac{a}{b}\right)^2 = 2$? This would imply that $a^2 = 2b^2$. (This means that $a^2 > b^2$.) Thus, a^2 is even, so a is even, so $a = 2c$ (where c is an integer), so $4c^2 = a^2 = 2b^2$, so $2c^2 = b^2$, so b^2 is even, so b is even. Thus, if $\frac{a}{b} = \sqrt{2}$, then both a and b are even. This is a problem, as we should be able to reduce the fraction. Now we can state and prove this formally.

Theorem: $\sqrt{2} \neq \frac{a}{b}$ for any integers a and b.

Proof: Assume $\frac{a}{b}$ is already in reduced form (else, reduce it). If $\sqrt{2} = \frac{a}{b}$, then $\frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is even $\Rightarrow a$ is even $\Rightarrow a = 2c$ for some integer $c \Rightarrow 4c^2 = a^2 = 2b^2 \Rightarrow b^2 = 2c^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even $\Rightarrow \frac{a}{b}$ is not reduced. This is a contradiction! Q.E.D. (quod erat demonstrandum, which means "that which had to be proved.") We can only reach a false statement making logical deductions from an initial claim if the initial claim was false, so we've proved the initial claim.

Final note: this proof uses symbols and words. Note that words can be ambiguous, so be careful!