Wednesday,
February 6

How should we start approaching a problem? Try plugging in numbers, plug in values for some variables but not others, refer to the class notes, et cetera. Don't stare at the ceiling until you've tried everything!

Last time, we proved $\sqrt{2}$ can't be written as a fraction.

Definition: We say a number is <u>rational</u> if it can be expressed in the form $\frac{a}{b}$ with a & b integers; we say a number is <u>irrational</u> if it's not rational.

Examples: 0 is rational, as $0=\frac{0}{1}$. 1.237 is rational, as $1.237=\frac{1237}{1000}$. $\sqrt{2}$ is irrational (we proved this). $\sqrt{3}$ is irrational (this is on the homework).

What about π? Professor Goldmakher said it's irrational, but look at the definition: "the ratio of the circumference of a circle to its diameter." How do we know that's true for all circles. It's not obvious — we'd probably have to use calculus, so we add "unit" before "circle" (which I accidentally did preemptively, as you can see by the scratch marks above). But, yes, π is irrational.

Another notable irrational number is e. What is e? We know $\ln$ is log base e. What is $\ln$? $\int x^3 dx = \frac{x^4}{4}+c$. What is $\int x^{-1} dx$? It's $\ln x + C$. It turns out that $\int x^{-1} dx$ is a log base some number, and we call that number e. (e is also defined as the only value of a such that $\int a^x dx = a^x + c$.)

It was proven relatively recently (by Lindemann in the 1870s) that e is irrational.

Wait a second — the 1870s are recent? Yes, in math terms. This leads us to two open questions:

1. Is $e+\pi$ irrational? 2. Is $e\pi$ irrational?

(Professor Goldmakher conjectures that both of these numbers are irrational. Oddly, it is known that e^{π} is irrational.)

Back to e: can we define it with the Taylor series? (We'll come back to Taylor series in a few weeks.)

Thus ends the discussion of rational numbers! One final note before we end, though — in future math classes, we'll prove that the rational numbers comprise 0% of the real number line. (This has some relation to differing sizes of infinity, which we'll get to later.)

Let's talk about prime numbers! What is a prime number?

After some refinement from the initial idea, we settled on:

Definition: A prime number is a positive integer whose only positive factors are 1 and itself.

But what's a factor? We reached the following definition:

Definition: Given integers n and d, we say d is a factor of n if $\frac{n}{d}$ is an integer.

A brief philosophical side note: what's the point? Is there such a thing as a prime? No. We made it up!

Example: what is 2? As an adjective, two makes sense (two chairs, two trees, 2 Chainz, etc.), but making it a noun is bizarre. However, even though mathematics is synthetic, it's crucial to life — people discover stuff that changes the world for fun, like George Boole (we'll get back to him, but he's crucial for computers).

Back to primes! Primes are very fundamental assets of math (think back to prime factorization trees from elementary school).

Question: How does 1 fit into our definition?

Answer: By our definition, it's prime. This isn't a good thing. If primes are supposed to be "building blocks" of numbers 1 isn't a great building block: $1 \cdot 1 \cdot \ldots \cdot 1 = 1$. (There's an xkcd comic about this.) Thus, our initial definition was flawed. Here's the definition we'll work with for the rest of the course:

Definition: An integer $n \geq 2$ is called "<u>composite</u>" if we write $n = ab$ for some integers a & b satisfying $1 < a, b < n$. An integer $n \geq 2$ is "<u>prime</u>" if it's not composite.

Q: What is the largest prime number?

Answer 1: Nobody knows.

Answer 2: There isn't one; i.e. primes go on forever.

Theorem: There are infinitely many prime numbers. (We'll get to that next class.)