Monday,
February 11

It's time for topic 2: Logic!

Q: What is a proof? (We've been doing proofs, but ~~[illegible]~~ what, exactly, is a proof?)

A: It depends on what you're talking about. (There are legal, philosophical, scientific, and mathematical proofs, for instance.)

Let's tackle a philosophical proof: Descartes's "Cogito ergo sum" (which means "I think, therefore I am"). While convincing, Descartes takes three steps from this statement to prove that ~~[illegible]~~ an infinitely powerful god exists, which seems like a bit of a stretch in such a small number of steps.

Let's go back and refine our original question, then:

Q: What is a mathematical proof? Let's try to define it.

It could be a justification for a conjecture, or showing a conclusion is necessarily true by using premises that we already know (or agree) to be true.

Eventually, Professor Goldmakher presented a definition:

Definition: A "<u>mathematical proof</u>" is a finite sequence of logical deductions starting from a collection of axioms and concluding in the statement of a proposition.

Unfortunately, this definition uses some words we haven't defined yet. Let's define one!

Definition: A "<u>proposition</u>" is a declarative statement that is either true or false.

This doesn't seem super restrictive, ~~[illegible]~~ Let's check out some examples to test this definition.

1. "Williams College is in Williamstown." This is a proposition — it's declarative, and it's either true or false,

2. "Leo is a frog." This is a proposition, as it's declarative and either true or false.
3. "It's hot in here." This is not a proposition as there's not an objective definition of hot, so it cannot be definitively true.
4. "Every even integer ≥ 4 is the sum of two primes." This is a proposition, as it's declaritive and either true or false. (Interestingly, we don't know whether it's true or false. This is Goldbach's conjecture, and while it's been tested a lot, it hasn't been ~~proven~~ proven true or false yet. (Also, Goldbach's conjecture was created by Euler, which leads to Goldmakher's Law: ~~that~~ nothing in mathematics is named after the discoverer. Humorously, this is a preexisting law known as Sturgeon's Law, which, even more humorously, wasn't discovered by Sturgeon!))
5. "This sentence is false."
Wait, what? If we suppose it's true, it ends up being false, but if we suppose it's false, it ends up being true... what?
It turns out that this isn't a proposition, as it's either both true and false or neither true nor false.
This leads us to:
<u>Boolean algebra</u>: an "arithmetic" of propositions via logical connectives: and, or, not, if... then...
Let's do some examples, but first, let's define some variables:
P = "Williams is in Williamstown." Q = "Leo is a frog."
1. Not (Q) = True, since Q is false.
2. P and Q = False, since Q is false.
3. P or Q = True, since P is true.

4. If P then Q. This is false. Since P is true, we don't want to get a false result, and since Q is false, the statement is false.

5. If Q then P.

Wait, what? Let's try to abstract this a bit.

If (true) then (false) is false – as above, we don't want to start with $0=0$ and use logical deduction to get $2=3$.

If (false) then (true). Suppose $2=3$. Then $2\times 0=3\times 0 \Rightarrow 0=0$, which suggests that the previous statement should be true.

Here's an example:

"If you give Professor Goldmakher a Lamborghini, then he'll give you an A."

- If you give him a Lamborghini and he gives you an A, the statement is true.
- If you don't and he gives you an A, the statement is true.
- If you don't and he doesn't give you an A, the statement is true.

The statement is only a lie if you give Professor Goldmakher a Lamborghini and he doesn't give you an A, which further suggests that "If Q then P" is true.