Monday,
February 18

There's an analogy between Boolean algebra and arithmetic. Why? There are four core operations that we can use to combine things (numbers or propositions).

We can make this connection literal! How? ~~We~~ By assigning numerical values to the truth values of propositions, say

$$\#P = \begin{cases} 1 \text{ if } P \text{ is true} \\ 0 \text{ if } P \text{ is false} \end{cases}$$

Now we can turn our truth tables into ~~arithmetic~~ arithmetic! Let's do an example

$\#P$	$\#Q$	$\#(P \wedge Q)$
1	1	1
1	0	0
0	1	0
0	0	0

This is a multiplication table! In other words, $\#(P \wedge Q) = \#P \cdot \#Q$.

(Note: our definition of $\#P$ is somewhat arbitrary. It comes out nicely when discussing $\wedge$, but, as will be shown on the homework, other values might work more nicely while discussing $\vee$.)

Question: What is $\#(\neg P)$?

Answer: $\#(\neg P) = 1 - \#P$:

$\#P$	$\#(\neg P)$
1	0
0	1

(This content will come up more on the homework, but for now, we're moving on.)

There's one more logical operator to discuss: the biconditional! But first, a question: what is a perfect square? (Numerically, not geometrically.)

A proposed definition: The square root of a perfect square yields an integer.

Is 5 a perfect square? By this definition, no, since $\sqrt{5}$ isn't an integer.
Is 4 a perfect square? With this definition, we can't tell. While $\sqrt{4}$ is an integer, we don't know if there are non-perfect squares with integer square roots from this definition.
To fix this, we introduce the biconditional $\Leftrightarrow$, defined:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \Leftrightarrow Q$ is read "P if and only if Q." This is a lot to write, so it's written "P iff Q."

Armed with this knowledge, we can revise our definition:
Definition: A number is a "perfect square" iff it's the square of an integer.
It turns out:

$$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

$$(\text{or } P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (P \Leftarrow Q))$$

(This makes sense conceptually, as the biconditional is an implication in two ways — we need "if P, then Q" and "if Q, then P" to have a biconditional.)
Let's try proving an if and only if statement.
Theorem: Given an integer n, n^2-1 is a perfect square iff $n = \pm 1$.
Proof:
($\Leftarrow$) If $n = \pm 1$, then $n^2 - 1 = 0$, which is a perfect square.
($\Rightarrow$) If n^2-1 is a perfect square, then $n^2 - 1 = a^2$ for some integer a.

$\Rightarrow n^2 - a^2 = 1$

$\Rightarrow (n-a)(n+a) = 1$

$\Rightarrow n-a = n+a = 1$ or $n-a = n+a = -1$

$\Rightarrow n-a = n+a$

$\Rightarrow a = 0$

$\Rightarrow n^2 - 1 = 0$

$\Rightarrow n = \pm 1.$ Q.E.D.