Wednesday,
February 20

Today we'll be wrapping up logic! (The problem with logic is that it's a lot of talking about math instead of doing math.)

First, let's talk about a special type of proposition: the predicate. Let's go back to Goldbach's conjecture:

Every even integer ≥ 4 is the sum of two primes.

(Remember, even though we don't know the truth value, this is still a proposition.)

This is a fancy proposition, though: it represents many propositions. This is an example of a predicate.

Definition: A "predicate" is a sentence that employs finitely many variables and becomes a proposition for any choice of those variables allowed by the sentence.

~~Even~~ Wait a second... we said Goldbach's conjecture was a predicate! Where are the variables? It turns out that the variables are the even integer and the two primes. Let's rewrite Goldbach's conjecture to make these variables explicit.

Goldbach's Conjecture, Revamped: For all even numbers $n \geq 4$, there exist primes p and q such that $n = p + q$.

In addition to variables, predicates employ quantifiers (like "for all" and "there exist") and relations (like "n is even" and "p is prime"). Let's rewrite Goldbach's conjecture again to make these clearer.

Goldbach's Conjecture and the Prisoner of Azkaban: For all n belonging to {even numbers ≥ 4}, there exist p, q belonging to {primes} such that $n = p + q$.

We've seen other predicates, too! What are they?

Let's take a trip down memory lane.

(1) $1+2+3+\dots+N = \frac{N(N+1)}{2}$ is a predicate:

For all N belonging to {positive integers}, $1+2+3+\dots+N = \frac{N(N+1)}{2}$.

(2) $\sqrt{2}$ is irrational is also a predicate:

For all a belonging to {integers} and all b belonging to {nonzero integers}, $\sqrt{2} \neq \frac{a}{b}$.

(Note: we need to specify b is nonzero, as otherwise, we could divide by zero, which isn't allowed.)

(Additional note: technically, even though we can write $\sqrt{2}$, that doesn't mean it exists. We can write $\sqrt{\text{orange}}$, for instance, but that doesn't mean $\sqrt{\text{orange}}$ exists. If we'd like to be very careful, then, we can rewrite $\sqrt{2} \neq \frac{a}{b}$ as $2 \neq \left(\frac{a}{b}\right)^2$.)

(3) Every $n \geq 2$ has a prime factor is a predicate:

For all n belonging to {integers ≥ 2}, there exists p belonging to {primes} such that n is divisible by p.

I don't know about you, but I'm getting tired of writing "for all," "there exists," and "belonging to." Luckily, there are symbols to shorten predicates!

"For all" $= \forall$

"There exists" $= \exists$

"Belonging to" $= \in$

Let's use these to write that there are infinitely many primes as a predicate.

(4) $\forall n \in \{\text{positive integers}\}, \forall p_1, p_2, \dots, p_n \in \{\text{primes}\}, \cancel{\exists q}$ ~~such~~ $\exists q \in \{\text{primes}\}$ such that $q \neq p_1, q \neq p_2, \dots q \neq p_n$.

Let's go back now to where this discussion started: what is a proof?

Let's talk about how math works for a moment.

How math works:

First, start with a finite collection of axioms (propositions that we define to be true). Next, using these axioms and logical connectives ($\land, \lor, \neg, \Rightarrow$), create other true propositions. More generally, we create theorems. (A theorem is a proposition that's a consequence of a finite logical deduction from propositions previously known to be true.) The set of all theorems arising from a given set of axioms is called a "<u>mathematical theory</u>."

(Note: theory has a different meaning here than in science. In math, theory is based on theorems that are known to be true. In science, we may have a theory that we try to prove.)

Examples:

(1) Euclidean geometry.

Euclid wrote down 10 axioms (the fifth, that if we have a line and a point not on the line, there's a line through the point that doesn't intersect the line, is probably the most famous). Everything we know about Euclidean geometry (the Pythagorean theorem, that the sum of the angles in a triangle is $180°$, etc.) comes from those axioms.

(2) Non-Euclidean geometry.

Remember that fifth axiom? What if that line that we said existed didn't exist? If we keep the other axioms and just change that one axiom, we get a whole new geometry!

(Note: Non-Euclidean geometry is the geometry of curved spaces, so it's probably more accurate when it comes to reality.)

③ Peano axioms:

There are nine. They determine how integers behave. (They're cool! Look them up!)

We end on this theorem.

Theorem (Gödel, 1933):

Any sufficiently interesting math theory cannot be both consistent (~~[illegible]~~ meaning the axioms don't lead to contradictions) and complete (meaning every true proposition is provable).