Friday
February 22

We're moving on to functions and sets!

What is a set?

Proposed definition: A set is a collection of things.

But what is a collection? It's tricky to pin down!

What are some sets we've seen?

The set of all integers, or {integers}. This is such an important set that we denote it $\mathbb{Z}$. (Why $\mathbb{Z}$? It's from "zahl," the German word for number.)

$\mathbb{Q}$ denotes the set of all rational numbers, or {rationals}.

$\mathbb{Z}_{\geq 0}$ denotes ~~integers $\mathbb{Z}_{\geq 0}$ or~~ the set of all integers ≥ 0, or {integers ≥ 0}.

(Sometimes, this ~~things~~ is called $\mathbb{N}$, the natural numbers.)

$\mathbb{Z}_{>0}$ denotes the set of all positive integers, or {positive integers}.

(Sometimes, this is called $\mathbb{N}$, the natural numbers.)

Wait, what? There's disagreement among mathematicians about whether 0 is a natural number, so we'll probably avoid using $\mathbb{N}$.

$\mathbb{R}$ denotes the set of all real numbers, or {real numbers}.

In this class, we'll be doing naive set theory, where we act like everyone knows what a set is. This can lead to some strange happenings, like the set of all sets S being an element of itself. To avoid these issues, Zermelo introduced a set of axioms of set theory. These became ZFC, a set of 9 axioms. (Note: the C stands for choice. We'll come back to that later.)

Now, let's talk about infinity. It's not a number! To discuss it, it's time for story time.

Story time: Hilbert Hotel

One day, NASA found a strange object in the Cantor galaxy. They didn't know what it was, so they sent astronaut Io Tichy to investigate. It turns out it was a hotel with an infinite number of rooms! All of the rooms had the standard appliances, like hot tubs and plasma decouplers. Io had been journeying to the hotel for a long time, so she was tired and wanted to stay in the hotel. Unfortunately, there was a math conference, so all of the rooms were occupied! Now, the rooms in the hotel are enumerated by $\mathbb{Z}_{>0}$, i.e. 1, 2, 3, 4, ... The manager tells Io not to fret, though! They can move the person in room 1 to room 2, room 2 to room 3, and, in general, room n to room $n+1$. This leaves room 1 available for Io!

After a night in the hotel, Io went to the lobby the next morning to find a group of mathematicians that arrived late (the holier-than-thou mathematicians from Hamster College). Unfortunately, all rooms are occupied, but, as we did before, we can move the person in room n to room $n+9956$. (I forgot to mention that there were 9956 late mathematicians — sorry!) This leaves the first 9956 rooms vacant for the Hamster College mathematicians. Hooray!

The next morning, Io comes to the lobby to find that there's a philately convention at the same time as the math conference, so there are infinitely many philatelists who need rooms! How can they all fit? Well, the manager can send the person in room n to room $2n$. This leaves all of the odd rooms vacant. There are an infinite number of vacant rooms, then, so every philatelist gets a room!

Now, the owner of the original Hilbert Hotel decided to open an infinite chain of Hilbert Hotels, each with infinitely many rooms. Unfortunately, this wasn't profitable, so he shut down all of them except the original. However, he guaranteed that everyone who was in a hotel that got shut down could get a room in the original hotel. How can we fit everyone?

Proposals:

- Move the person in room n to room n^2... but then how do we distribute the new guests?
- Move the person in room n to room n^n... but we hit the same problem.
- Put the person from room n in hotel h into room n^h... but we put some people in the same room (like room 2 from hotel 2 and room 4 from hotel 1).

Professor Goldmakher proposed:

Send the person from room n in hotel h to room $2^n 3^h$.

Does this send multiple people to the same room?

Suppose we do. Then there's some room a in hotel b that gets sent to the same room as room c in hotel d. This means $2^a 3^b = 2^c 3^d \Rightarrow 2^{a-c} = 3^{d-b}$. The left hand side of this is even unless $a-c=0$, in which case $2^{a-c} = 2^0 = 1$. Thus, if $2^{a-c} = 1 = 3^{d-b}$, $a-c=0$ and $d-b=0 \Rightarrow a=c$ and $b=d$, so the room a in hotel b was the same as the room c in hotel d. They were the same person! No two people get sent to the same room! Hooray!

Next time, we'll talk about different sizes of infinity.