Monday, February 25

Before we begin, a brief aside. Consider the following:
"If you're in the ocean, then you're not in water."
(This is false.)
"If you're not in water, then you're in the ocean."
(This is also false.)
Wait, what?
This is a statement and its converse. A truth table tells us that these should never be false at the same time, so what's going on?

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$
T	T	T	T
F	T	T	F
T	F	F	T
F	F	T	T

Well, "you're not in water" feels a little iffy, but there's something more concrete: neither of these are propositions. They're actually predicates, and a document will be posted online later today diving into this topic further.

Let's go back to the Hilbert Hotel!
Recall that last time, given (h, n) (where h represents which Hilbert Hotel they came from and n represents their room number), we can send that person to room $2^h 3^n$ in the original hotel. This sends everyone to a different room in the original hotel, but this leaves a lot of empty rooms (for instance, rooms 5, 7, and 11 won't be occupied).
Is there any way to be more efficient?
Let's try visualizing every guest from every Hilbert Hotel.
To do this, let's construct a 2D array of the guests:
(on next page)

(1, 1) (1, 2) (1, 3) (1, 4)...
(2, 1) (2, 2) (2, 3) (2, 4)...
(3, 1) (3, 2) (3, 3) (3, 4)...
(4, 1) (4, 2) (4, 3) (4, 4)...
⋮ ⋮ ⋮ ⋮

We shouldn't go along the first row and put (1, 1) in room 1, (1, 2) in room 2, and so on, as we'd never get to (2, 1). But what if we do a diagonal argument?

(1 1 1) (1 3 2) (1 6 3) ... (and so on)
(2 2 1) (2 5 2) (2, 3) ...
(3 4 1) (3, 2) (3, 3) ...
⋮ ⋮ ⋮

This gives us an algorithm for placing each guest in a room of the original Hilbert Hotel so that no two guests share a room and every room is occupied. (There is a formula for this diagonal argument but that's left as a challenge.)

The above diagonal argument ~~gives us a fee~~ seems to give us the intuition that the original Hilbert Hotel has the same "size" as all of them. How can we compare "sizes" of different infinite sets A and B?

Let's back up for a second. Let's say there's a group of people outside of a theater and a collection of seats inside. How can we compare the number of people to the number of seats? We could count them... or we could open the theater. This leaves three outcomes: there are empty seats remaining, there are people without seats, or all seats are taken and everyone is sitting.

These tell us, respectively, that there were more seats than people, there were more people than seats, or there were an equal number of seats and people. This leads us to Cantor's idea: We say two sets A and B have the same "cardinality" (a fancy word for size) if and only if there exists a one-to-one correspondence between the elements of A and the elements of B.

Example: $\mathbb{Z}_{>0} = \{1, 2, 3, 4, \ldots\}$

vs $2\mathbb{Z}_{>0} = \{2, 4, 6, 8, \ldots\}$ $n \mapsto 2n$ gives a one-to-one correspondence.

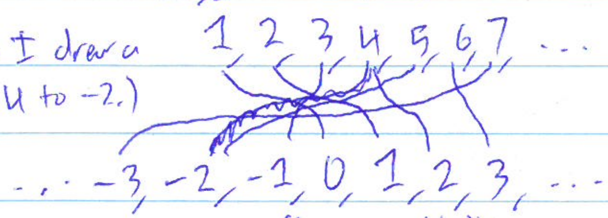
Example: $\mathbb{Z}_{>0} = \{1, 2, 3, 4, \ldots\}$

vs. $2\mathbb{Z}_{>0} + \frac{1}{2} = \{2.5, 4.5, 6.5, 8.5\}$

$n \mapsto 2n + \frac{1}{2}$ gives a one-to-one correspondence. ($\mapsto$ means "maps to" or "is sent to.")

What about $\mathbb{Z}_{>0}$ and $\mathbb{Z}$? Let's draw a picture!

(sorry... I drew a line from 4 to −2.)



Let's send 1 to 0, 2 to 1, 3 to −1, 4 to 2, and so on!

Definition: A set is "<u>countable</u>" if and only if it has the same cardinality as $\mathbb{Z}_{>0}$.

We've said "one to one correspondence" several times. What does that mean?

Definition: A one-to-one correspondence between sets A and B is a function f from A to B such that for all b in B, there exists a unique a in A such that $f(a) = b$.