Wednesday, February 27

Make sure you review your notes after class! Don't just read them — try to reproduce them.

What ~~[illegible]~~ is a countable set?

Definition: A set is countable if and only if there exists a one-to-one correspondence between that set and $\mathbb{Z}_{>0}$.

What is a one-to-one correspondence?

Definition: A one-to-one correspondence between sets A and B is a function $f: A \to B$ such that ~~$\forall b \in B$, unique $a \in A$~~ for all b in B, there exists a unique a in A such that $f(a) = b$.

There's some new notation in that definition. Let's define it!

$f: A \to B$ a function means f associates to each a in A a unique b in B, which we denote $f(a)$ (i.e. $b = f(a)$).

Example: $f: \mathbb{Z} \to \mathbb{R}$ defined by $f(x) = x^2$.

Notes: • We're not allowed to plug $\frac{1}{2}$ into this function. We can square $\frac{1}{2}$, but $\frac{1}{2} \notin \mathbb{Z}$, so it's not a valid input.

- $-1 \in \mathbb{R}$, but -1 is never an output of f. (There are other elements of $\mathbb{R}$ that this is true for, too!)
- $f(-2) = f(2) = 4$. This is okay! Our definition says it's okay for two different inputs to return the same output, but one input cannot return two outputs.

Non-Example: $g: \mathbb{Z} \to \mathbb{R}$, where $g(n)$ is the solution x to $x^2 - n = 0$. This is not a function!

- $g(4) = 2$ or -2, as $2^2 - 4 = (-2)^2 - 4 = 0$. This is unacceptable!
- $g(-1)$ isn't a real number (no real number solves $x^2 + 1 = 0$). This is also unacceptable!

Going back to f, we can write it as $x \longmapsto x^2$. This demonstrates the difference between $\rightarrow$ and $\mapsto$: $\rightarrow$ denotes a function goes from one set to another, while $\mapsto$ denotes what to do to a particular element of a set.

Functions don't have to use numbers!

Example: $h\{\text{animals}\} \rightarrow \{0, 1\}$.

$$h(x) = \begin{cases} 1 & \text{if } x \text{ can breathe underwater} \\ 0 & \text{otherwise} \end{cases}$$

For example, $h(\text{humans}) = 0$.

Let's return to countability! Here's an intuitive way to think about countability:

There exists a way to count off all the elements (i.e. first, second, third, etc.) in such a way that any element is reached within a finite time.

For example, $\mathbb{Z}$ is countable:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

third first second fourth...

What about $\mathbb{Q}_{>0}$? Let's write fractions as an array:

$$\begin{array}{cccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} \; \ldots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} \\ \vdots & & & \end{array}$$

Let's cross off unreduced fractions:

$$\begin{array}{cccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} \\ \frac{1}{2} & \cancel{\frac{2}{2}} & \frac{3}{2} & \cancel{\frac{4}{2}} \; \ldots \\ \frac{1}{3} & \frac{2}{3} & \cancel{\frac{3}{3}} & \frac{4}{3} \\ \frac{1}{4} & \cancel{\frac{2}{4}} & \frac{3}{4} & \cancel{\frac{4}{4}} \\ & \vdots & & \end{array}$$

Now let's do the diagonal argument we did with Hilbert hotels!

$$\begin{array}{cccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} \\ \frac{1}{2} & \cancel{\frac{2}{2}} & \frac{3}{2} & \cancel{\frac{4}{2}} \; \ldots \\ \frac{1}{3} & \frac{2}{3} & \cancel{\frac{3}{3}} & \frac{4}{3} \\ \frac{1}{4} & \cancel{\frac{2}{4}} & \frac{3}{4} & \cancel{\frac{4}{4}} \\ & & \vdots & \end{array}$$

Therefore, $\mathbb{Q}_{>0}$ is countable.

Well, then... what isn't countable?

Theorem (Cantor): The set $(0,1)$ consisting of all real numbers x such that $0<x<1$ is not countable.

Proof strategy: Given <u>any</u> enumeration of $(0,1)$, I'll construct an $\alpha \in (0,1)$ that the enumeration will <u>never</u> list.

Proof: Pick any enumeration of $(0,1) = \{x_1, x_2, x_3, x_4, \ldots\}$.

For example, $x_1 = 0.2309158\ldots$

$x_2 = 0.555260000000\ldots$

$x_3 = 0.12093762\ldots$

and so on.

Let $\alpha = 0.a_1a_2a_3a_4\ldots$, where a_1 is different from the first digit of x_1 (for instance, in our example, a_1 could be 1), a_2 is different from the second digit of x_2 (for instance, in our example, a_2 could be 1), a_3 is different from the third digit of x_3 (for instance, in our example, a_3 could be 7), and so on. More generally, take a_n to be different from the n^{th} digit of x_n for all $n \in \mathbb{Z}_{>0}$ (for instance, in our example, $\alpha = 0.117\ldots$). Note that $\alpha \neq x_1$, as they differ in the first digit, $\alpha \neq x_2$, as they differ in the second digit, and so on. For all $n \in \mathbb{Z}_{>0}$, $\alpha \neq x_n$. Thus, α was never enumerated. Q.E.D.