Friday, March 8

The midterm is now online! Have fun!

Recall: $f: A \hookrightarrow B \Leftrightarrow f$ is injective.

(Injective means everything in B comes from at most one input from A.)

$f: A \twoheadrightarrow B \Leftrightarrow f$ is surjective.

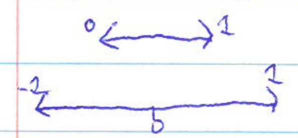
(Surjective means everything in B comes from at least one input from A.)

$f: A \hookrightarrow\!\!\!\!\rightarrow B \Leftrightarrow f$ is "<u>bijective</u>."

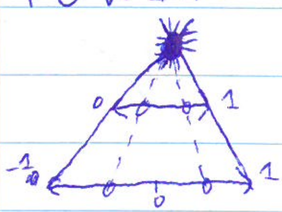
(Note: A bijective function isn't a one-to-one correspondence. It establishes a one-to-one correspondence.)

Proposition: $(0,1) \approx (-1,1)$.

How would we go about showing this? We can't do much with these, as we can't conceptualize what's in these sets (remember, we can't enumerate $(0,1)$). We can visualize them geometrically, though:



So what do we do? WE HARNESS THE POWER OF THE SUN!



It looks like any line from our sun through $(0,1)$ passes through $(-1,1)$ and any line from our sun through $(-1,1)$ passes through $(0,1)$, but this isn't a proof!

We need a function.

Proposed function: $f:(0,1) \to (-1,1)$

$$x \longmapsto 2x-1$$

~~Let's prove it,~~ This looks promising. Let's prove it!

Proof: Consider $f:(0,1) \to (-1,1)$

$$x \longmapsto 2x-1$$

Step 1: f is a function.

Given $x \in (0,1) \Rightarrow 0<x<1 \Rightarrow 0<2x<2 \Rightarrow -1<2x-1<1 \Rightarrow f(x)=2x-1 \in (-1,1)$. //

Step 2: f is injective.

$$f(x)=f(y) \Rightarrow 2x-1=2y-1 \Rightarrow x=y. \text{ //}$$

Before step 3, it's time for wishful thinking.

Want: $2x-1=f(x)=y \Rightarrow x=\frac{y+1}{2}$

Step 3: f is surjective.

Pick an arbitrary $y \in (-1,1)$. Then $f\left(\frac{y+1}{2}\right)=y$. However, we must check that $\frac{y+1}{2} \in (0,1)$: $y \in (-1,1) \Rightarrow -1<y<1 \Rightarrow 0<y+1<2 \Rightarrow 0<\frac{y+1}{2}<1$. //

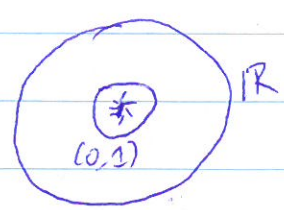
Q.E.D.

Our intuition for uncountable sets isn't great. For instance, it looks like $(-1,1)$ should be bigger than $(0,1)$ - it contains $(0,1)$ and is twice as long. (Side note: length is weird - how do we truly measure things?) An example of this intuition is the Banach-Tarski Paradox, which states that a solid sphere can be broken into five components, then reassembled into two spheres that are solid and the same size as the initial sphere. (This is true!)

Proposition: $(0,1) \cong \mathbb{R}$.

How can we go about this? We can't stretch $(0,1)$ by a constant like we did before, as no constant will encompass all of $\mathbb{R}$. What do we do?

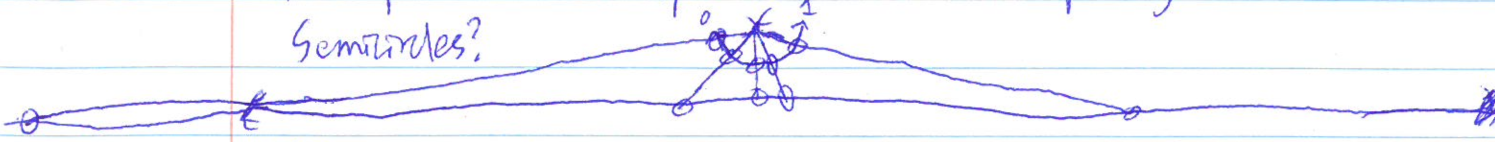
Circles?



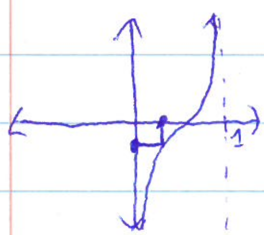
This could work... but how big is the outer circle? Where do they meet up with themselves?

Spirals?

How do we make elements of $(0,1)$ correspond to the surface of the sphere? How do we account for the points on the sphere that don't map anywhere? Semicircles?

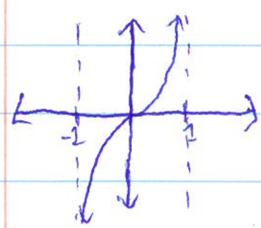


This works! At first, we may be concerned that 0 and 1 aren't sent anywhere, but 0 and 1 aren't on the interval! There's another way, though:



The inputs of this are elements of $(0,1)$, and the outputs are real numbers!

Another visualization:



We can get a function from this: $f(x) = \frac{x}{1-x^2}$. This gives us a bijection between $(-1,1)$ and $\mathbb{R}$ (in other words, $(-1,1) \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}$). Since $(-1,1) \hookrightarrow\!\!\!\!\rightarrow (0,1)$, with a bit of work, we can show $(0,1) \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}$ by getting a function sending an element of $(0,1)$ to an element of $\mathbb{R}$, and we can show that this function is bijective.