Monday, March 11

DO NOT USE ON MIDTERM

Let's see a set theory power tool (~~that~~ that you <u>cannot use on the midterm</u>):

Theorem (Cantor–Schröder–Bernstein) (proved by Dedekind): If $A \hookrightarrow B$ and $B \hookrightarrow A$, then $A \approx B$.

(The intuition makes sense – if A is at least as big as B and B is at least as big as A, they should have the same size.)

This is surprisingly hard to prove! We're not going to prove it.

(Note: Cantor wrote it down, Dedekind proved it, Schröder came up with an incorrect proof, and Bernstein came up with a different proof.)

This is useful! Let's do an example:

Claim: $\underbrace{[0,1]}_{\{x \in \mathbb{R} : 0 \le x \le 1\}} \approx (0,1)$.

Proof: $(0,1) \hookrightarrow [0,1]$

$x \longmapsto x$

and $[0,1] \hookrightarrow (0,1)$

$x \longmapsto \frac{x+1}{x+2}$ or $\frac{x}{10} + \frac{1}{2}$.

Thus, $[0,1] \approx (0,1)$ by C–S–B. Q.E.D.

It's time for our last set theory/functions topic!

What if $A \not\hookrightarrow B$?

Intuitively: A is strictly larger than B.

Formally: $\forall f : A \to B$, $\exists b \in B$ s.t. $f(a) = b = f(a')$ for some $a \neq a'$.

This is called the pigeonhole principle. (Editor's note: I absolutely love the pigeonhole principle.)

Colloquially, this says that if there are more pigeons than pigeonholes, then one of the pigeonholes has at least two pigeons in it.

Let's do some examples!

① Your sock drawer contains a bunch of black, white, and blue socks. How many socks do you need to remove to guarantee a pair? Four. What are the pigeons and pigeonholes?

Pigeons: Socks.

Pigeonholes: Colors — black, white, blue.

By drawing four socks, we guarantee that we have two pigeons in the same hole. Here, having two pigeons in the same hole means we have two socks of the same color. That's a pair! (Note: we don't know that the first three socks are all different colors. We could have drawn three blue socks first, so we already would have had a pair by the time we draw the fourth sock. What we do know, though is that we must have at least one pair after drawing four socks.)

② I picked 51 distinct integers out of the set $\{1, 2, 3, \ldots, 100\}$. Prove that two of my numbers are consecutive.

What are the pigeonholes?

Are they even and odd numbers? No... having two even or two odd numbers doesn't tell us that we have two consecutive numbers.

What about $1, 3, 5, 7, \ldots, 99$? No... what do we do if we chose 8?

What about $\{1,2\}, \{3,4\}, \ldots, \{99,100\}$? This looks good! We have 50 pigeonholes. Our pigeons are the 51 numbers, so one pigeonhole must have two pigeons (since $51 > 50$). One pigeonhole having two pigeons means we have two consecutive numbers, so we've proved it!

③ Prove that out of any 5 points on the surface of a sphere, 4 of them must lie on a hemisphere.

We could choose three of our points and construct a plane. However, which three do we choose? What if there's an evil configuration?

Here's how we do it:

see below for alternative formulation

Pick any two points, x and y. Connect x and y by a geodesic (the path of shortest distance along the surface of the sphere). Extending this path creates an equator. It splits the sphere into two hemispheres. x and y both lie on this equator. We have three points and two hemispheres. Now it's pigeonhole time! One of these hemispheres must contain two of the remaining points. Q.E.D.

Another way to think about this: pick two of the points x and y as above. There's a unique plane containing the three points x, y, and the center of the sphere. This plane splits the sphere into two hemispheres, with x and y lying on an equator. Now continue as above!