Wednesday,
March 13

Reminder: we do not have class this Friday!
Don't be too concerned about grades just yet. If you want to drop the class or switch to pass/fail, talk to Professor Goldmakher.
There is a problem set due the Thursday after break.
However, there is not a quiz the Monday after break.

Let's talk about induction...but first, a demonstration.
After the demonstration, note that all of the dominoes fall down if each domino knocks the next one down and the first domino falls. This leads us to:

Theorem (Induction): Suppose $A \subseteq \mathbb{Z}_{>0}$ satisfies
1. $1 \in A$
2. Whenever an integer lives in A, the next largest integer also lives in A.

Then $A = \mathbb{Z}_{>0}$.
(Notice the similarities to the demonstration: $1 \in A$ is similar to the first domino falling, and condition 2 is similar to each domino knocking down the next domino.)
We can rewrite condition 2: $n \in A \Rightarrow n+1 \in A$.
We can prove this theorem (and we will)...but why do we care?
Let's do some examples.
(1) Proposition: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$ $\forall n \in \mathbb{Z}_{>0}$. (We've already proved this, but now we'll prove it in a different way!)
Proof: By induction.
Let $A = \{n \in \mathbb{Z}_{>0} : 1+2+\ldots+n = \frac{n(n+1)}{2}\}$.
$1 \in A$ because $1 = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$. ✓
If $k \in A$, then $1+2+\ldots+k = \frac{k(k+1)}{2} \Rightarrow 1+2+\ldots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) =$
$(k+1)\left(\frac{k}{2}+1\right) = (k+1)\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2} \Rightarrow k+1 \in A$. By induction, $A = \mathbb{Z}_{>0}$. Q.E.D.

A brief aside: induction is <u>weird</u>. In Professor Goldmakher's experience, 50% of students get it right away and 50% of students see it as voodoo. Let's try to demystify the voodoo and keep it interesting.

The underlying principle of proof 1 is this:

Question: What is $1^3+2^3+\dots+10^3$?

Hint: $1^3+2^3+\dots+9^3=2025$.

This hint ~~tells us~~ helps us see that we just need $2025+10^3=3025$.

This is induction: knowing about one case helps us figure out the next one.

(By the way, 2025 and 3025 are perfect squares. This will come back on the homework.)

(Also, fun note: if a number is of the form $10n+5$, $(10n+5)^2 = 100(n)(n+1)+25$. For instance, $45^2=2025$ and $35^2=1225$.)

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2 Proposition: $n<4^n \ \forall n \in \mathbb{Z}_{>0}$.

Proof: By induction. Let $A=\{n\in\mathbb{Z}_{>0} : n<4^n\}$.

$1\in A$ because $1<4^1=4$.

Now it's time for some wishful thinking.

Scratchwork (not part of proof!)

Want: $k\in A \Rightarrow k+1\in A$.

$k\in A \Rightarrow k<4^k \Rightarrow k+1<4^k+1$.

Want: $4^k+1<4^{k+1} \Leftrightarrow 1<4^{k+1}-4^k=4^k(4-1)=3\cdot 4^k \Leftrightarrow \frac{1}{3}<4^k$

If $k\in A$, then $k<4^k$. Also, since $k>0$, $4^k>1>\frac{1}{3} \Rightarrow 3\cdot4^k>1 \Rightarrow (4-1)4^k>1$ $\Rightarrow 4\cdot4^k-4^k>1 \Rightarrow 4^{k+1}-4^k>1 \Rightarrow 4^{k+1}>4^k+1>k+1 \Rightarrow k+1\in A$.

By induction, $A=\mathbb{Z}_{>0}$. Q.E.D.

Now, we've got a bit of time left. Let's prove induction!

Proof of induction:

Given A satisfying the hypotheses of induction, suppose $A \neq \mathbb{Z}_{>0}$. Then $B = \mathbb{Z}_{>0} \setminus A \neq \emptyset$.

Let α be the smallest element of B. Note $\alpha \neq 1$ because $1 \in A \Rightarrow 1 \notin B$. Also, $\alpha - 1 \in A$ because α is the smallest integer not in A. (This secretly also uses that $\alpha \neq 1$, as if $\alpha = 1$, $\alpha - 1 = 0$, and since $A \subseteq \mathbb{Z}_{>0}$, if α were 1, $\alpha - 1 \notin A$, but we don't have to worry about that.) By condition 2, since $\alpha - 1 \in A$, $\alpha \in A$. However, this contradicts $\alpha \in B$ (hence $\alpha \notin A$). Q.E.D.

Final note: implicit in this proof is an axiom:

Well-Ordering Principle: Any non-$\emptyset$ subset of $\mathbb{Z}_{>0}$ has a least element.

Have a great break!