Monday,
April 1

We will go through the midterms on Wednesday.
Let's review (and continue) induction!
First, let's warm up with an example about $n!$ ("n factorial" – the product of all integers between 1 and n (for example, $3! = 3 \cdot 2 \cdot 1 = 6$). We aren't just super excited about n!)
(Side note: We have an explicit formula for the analogue of this with a sum instead of a product: $1+2+\ldots+n = \frac{n(n+1)}{2}$. We call it T_n, the n^{th} "<u>triangular number</u>": $T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$ $\ldots$)
~~Claim~~: $n! < 0$ $\forall n \in \mathbb{Z}_{>0}$.
"Proof": By induction. Let $A := \{n \in \mathbb{Z}_{>0} : n! < 0\}$. If $k \in A \Rightarrow k! < 0 \Rightarrow (k+1)! = (k+1)k! < 0 \Rightarrow k+1 \in A$. By induction, "Q.E.D."
Wait a second! We said above that $3! = 6$, and $6 \geq 0$. What's going on? Well...we forgot to check the base case. (The base case fails, by the way: $1! = 1 \geq 0$.) So, remember:
Theorem (induction): If $A \subseteq \mathbb{Z}_{>0}$ satisfies
(1) $1 \in A$ and
(2) $n \in A \Rightarrow n+1 \in A$,
then $A = \mathbb{Z}_{>0}$.
Let's do a real example now.
Claim: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$ $\forall n \in \mathbb{Z}_{>0}$.
Proof: By induction. Let $A := \{n \in \mathbb{Z}_{>0} : 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2\}$.
• $1 \in A$: $1 < 2$.
• Suppose $k \in A$. Then $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} < 2 \Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 + \frac{1}{(k+1)^2}$.
Well...where do we go from here? We're not sure! However, we might have an alternative proof not by induction. (We'll come back to induction.)

We know $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}<2 \ \forall n\in\mathbb{Z}_{>0}$. Then:

~~$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}} > 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$.~~

While this looks true for the initial terms, later terms in the first sum are actually smaller than those in the second (if $n=7$, $\frac{1}{2^6}=\frac{1}{64}<\frac{1}{49}=\frac{1}{7^2}$), so this falls apart. ☹ED.

Let's go back to induction. It fails, but it turns out that a stronger claim is true!

Claim: $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\le 2-\frac{1}{n} \ \forall n\in\mathbb{Z}_{>0}$.

Proof: By induction. Let $A:=\{n\in\mathbb{Z}_{>0}: 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\le 2-\frac{1}{n}\}$.

- $1\in A$: $1\le 2-\frac{1}{1}$. ✓
- Suppose $k\in A \Rightarrow 1+\frac{1}{2^2}+\ldots+\frac{1}{k^2}\le 2-\frac{1}{k}$. It's wishful thinking time!

  Want: $2-\frac{1}{k}+\frac{1}{(k+1)^2}\le 2-\frac{1}{k+1} \Rightarrow \frac{1}{(k+1)^2}\le\frac{1}{k}-\frac{1}{k+1}=\frac{k+1-k}{k(k+1)}=\frac{1}{k(k+1)}$.

  Indeed, $\frac{1}{(k+1)^2}\le\frac{1}{k(k+1)}$.

  $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2}\le 2-\frac{1}{k}\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2}\le 2-\frac{1}{k}+\frac{1}{(k+1)^2}\le 2-\frac{1}{k}+\frac{1}{k(k+1)}=$
  $2-\frac{1}{k}+\left(\frac{1}{k}-\frac{1}{k+1}\right)=2-\frac{1}{k+1}\Rightarrow k+1\in A$.

By induction, $A=\mathbb{Z}_{>0}$. Q.E.D.