Friday, April 5

Basic idea of induction: deduce a harder case from a simpler case that we already understand.

We proved $1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}<2$ for all integers n.

We COULD test it for $n=1$, $n=2$, $n=3$, and so on, but we cannot test it for every value of n manually (there are infinitely many values of n). Induction lets us boil down all of this manual testing into two steps: proving a base case and showing that one case being true implies the next case is also true.

It's time for strong induction!

<u>Theorem (Strong Induction)</u>

If $A \subseteq \mathbb{Z}_{>0}$ such that

① $1 \in A$ and

② $\{k \in \mathbb{Z}_{>0} : k<n\} \subseteq A \Rightarrow n \in A$,

then $A = \mathbb{Z}_{>0}$.

(Part 2 looks a bit funny. In simpler terms, it's just saying that if everything less than n is in A, then n is in A.)

There's an interesting side note here based on intuition from ① and ②:

$1 \in A \Rightarrow \{1\} \subseteq A \Rightarrow 2 \in A \Rightarrow \{1,2\} \subseteq A \Rightarrow 3 \in A \Rightarrow \{1,2,3\} \subseteq A \Rightarrow 4 \in A \Rightarrow \dots$

We're getting ahead of ourselves, though! Before we prove strong induction, let's do an example.

<u>Theorem</u>: Every $n \in \mathbb{Z}_{>0}$ can be expressed as a sum of distinct powers of 2 (or every $n \in \mathbb{Z}_{>0}$ can be written in binary).

Examples: $25 = 16+8+1 = 2^4+2^3+2^0$. $99 = 64+32+2+1 = 2^6+2^5+2^1+2^0$.

$8 = 2^3$ (there's no sum, but we count it). (Note: to write 25 in binary, we have 1 2^0, 0 2^1, 0 2^2, 1 2^3, and 1 2^4, which yields 11001, where the leftmost place is the 2^4 place the rightmost place is the 2^0 place. Likewise, 99 in binary is 1100011.)

Proof: By strong induction. Let $A := \{n \in \mathbb{Z}_{>0} : n \text{ can be written in binary}\}$.

- $1 \in A$: $1 = 2^0$ ✓
- Given $n \in \mathbb{Z}_{>0}$, suppose $\{k \in \mathbb{Z}_{>0} : k < n\} \subseteq A$. ($1 \in A$, so we may assume $n > 1$.)

It's wishful thinking time!

Want: $n \in A$.
Know: all numbers less than n can be written in binary, i.e. as a sum of distinct powers of 2.
Consider $n = (n-1)+1$. If n is odd, $n-1$ is even, and $(n-1)+1$ gives the binary for n. If n is even, this doesn't work, as we get two ones.
Uh oh!

We know n is either even or odd. If n is even, then $n = 2a$, and if n is odd, then $n = 2a+1$ for some $a \in \mathbb{Z}_{>0}$.

Note: $a \leq \frac{n}{2} \Rightarrow a < n$. Thus, $a \in A \Rightarrow a = 2^{e_1} + 2^{e_2} + \dots + 2^{e_\ell}$, where $e_i \in \mathbb{Z}_{\geq 0}$ for all i and all e_i are distinct.

If $n = 2a$:

$$n = 2a = 2(2^{e_1} + 2^{e_2} + \dots + 2^{e_\ell}) = 2^{e_1+1} + 2^{e_2+1} + \dots + 2^{e_\ell+1} \Rightarrow n \in A.$$

If $n = 2a+1$:

$$n = 2a+1 = 2(2^{e_1} + 2^{e_2} + \dots + 2^{e_\ell}) + 1 = 2^{e_1+1} + 2^{e_2+1} + \dots + 2^{e_\ell+1} + 2^0 \Rightarrow n \in A.$$

By strong induction, $A = \mathbb{Z}_{>0}$. Q.E.D.

Now, let's prove strong induction.

Proof of Strong Induction:

By induction.

Given A satisfying ① and ② from strong induction, we want $A = \mathbb{Z}_{>0}$.

Let $\mathbb{B} := \{L \in \mathbb{Z}_{>0} : \{\ell \in \mathbb{Z}_{>0} : \ell \leq L\} \subseteq A\}$.

- $1 \in \mathbb{B}$ because $1 \in A$.

Suppose $t \in \mathbb{B} \Rightarrow \{k \in \mathbb{Z}_{>0} : k \leq t\} \subseteq A \Rightarrow \{k \in \mathbb{Z}_{>0} : k < t+1\} \subseteq A \Rightarrow$ $t+1 \in A \Rightarrow \{1, 2, \ldots, t+1\} \subseteq A \Rightarrow t+1 \in \mathbb{B}$. By induction, $\mathbb{B} = \mathbb{Z}_{>0}$ $\Rightarrow A = \mathbb{Z}_{>0}$. Q.E.D.