AN EXPLICIT FORMULA FOR FIBONACCI NUMBERS

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1. Introduction

At the heart of induction is the idea that to prove a predicate, it suffices to be able to reduce any particular case of the predicate to a simpler case. Similarly, a recurrence relation is a way of defining a function by its previous behavior. The most famous example of this is the Fibonacci sequence:

$$\underbrace{1}_{f_1}, \underbrace{2}_{f_2}, \underbrace{3}_{f_3}, \underbrace{5}_{f_4}, \underbrace{8}_{f_5}, \underbrace{13}_{f_6}, \underbrace{21}_{f_7}, \underbrace{34}_{f_8}, \underbrace{55}_{f_9}, \underbrace{89}_{f_{10}}, \ldots$$

How do we express this formally?

$$f_{n+1} := f_n + f_{n-1} \quad \forall n \geq 2, \text{ where } f_1 := 1 \text{ and } f_2 := 2.$$

Historically, this was introduced by Leonardo, son (figlio) of Bonacci (aka 'Fibonacci'). Fibonacci's single greatest contribution was to popularize Hindu-Arabic numerals (which are vastly better than Roman numerals for doing arithmetic) in Europe. But his name is mostly associated to this sequence, which came from an odd example of a mathematical model for the growth of rabbit populations. His model went like this:

- rabbits take one month to reach maturity
- rabbits never die
- rabbits are monogamous
- a pair of mature rabbits conceive once per month
- gestation takes one month (i.e. from conception to birth takes one month)
- the brood always consists of one male and one female rabbit, and
- brother and sister become a mating pair.

In his book *Liber abbaci* (1202), Fibonacci posed the question: given the above model and starting with a single pair of bunnies, how many rabbits will there be at the end of a year? (Month 1: 1, Month 2: 2, Month 3: 3, Month 4: 5, ..., Month 12: 233) As a mathematical model, this turns out to be pretty terrible: almost all of its assumptions are incorrect, and it doesn't fit the data. The reason the sequence didn't fall into total obscurity is because it turns out to have a remarkable amount of structure, and seems to arise in nature pretty frequently.

In particular, the famous astronomer Johannes Kepler (1571-1630) became pretty obsessed with the Fibonacci sequence. He made a number of interesting observations about the sequence, including the fact that the ratio of consecutive terms tends to a number now called the *golden ratio*:

Conjecture 1.1 (Kepler)**.** $\lim_{n\to\infty} \dfrac{f_{n+1}}{f_n} = \dfrac{1+\sqrt{5}}{2}$

The golden ratio plays an important historical role in art and architecture, from the shape of the Parthenon and the theater of Epidaurus, to the works of Leonardo da Vinci and other renaissance artists, to the musical compositions of Bach and Bartók.

There are many proofs of Kepler's conjecture. You'll see a short and unexpected proof on your upcoming homework (using 'continued fractions'). When you take linear algebra (math 250) you might see a different proof using matrices. Today we'll deduce Kepler's conjecture from a much stronger result: an explicit formula for f_n.

I personally can never remember the precise statement of the formula. Instead, I remember the approach, and can work out the formula whenever I need it. More importantly, the approach we'll take – using something called a *generating function* – is very robust and applies to many problems about sequences and combinatorics. Let's go!

2. The generating function of the Fibonacci sequence

We want to study a neverending sequence of terms, which is hard to do. Instead, we combine all these terms into one single object that we study, called a generating function:

$$f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

The idea? Find the function $F(x)$ this is the Taylor series of, and then figure out a different way to derive its Taylor series. Before we can play with this, what's f_0? We can work it out: if we expect each term of the sequence to be the sum of the previous two, and $f_1 = 1$ and $f_2 = 2$, then f_0 must be 1. Let's write this generating function down more explicitly:

$$F(x) := 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots$$

There are lots of things we can do: differentiate or integrate both sides, plug in some values of x, or multiply by something. If we multiply both sides by x, we get

$$xF(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots$$

No surprises so far. But now consider

$$F(x) - xF(x) = 1 + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + \cdots$$

Anything jump out? There's a secret $F(x)$ on the right hand side!

$$F(x) - xF(x) = 1 + x^2 F(x).$$

As usual, we want to isolate the most mysterious thing here, which in this case is the $F(x)$. Solving we get

$$F(x) = \frac{1}{1 - x - x^2}.$$

In other words, we've just discovered that the Taylor series of this function has precisely the Fibonacci coefficients:

$$\frac{1}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots$$

The advantage of this is that the function on the right is explicitly about the Fibonacci numbers, while the function on the left has *nothing* to do with them – we can study it even without knowing anything about the Fibonacci numbers! So, let's do that.

3. Partial Fractions

In calculus, if you had some ugly fraction like this, one technique you learned about was partial fraction decomposition. The idea:

- factor the denominator, and
- split the fraction up as a sum of fractions whose denominators are the factors.

In our example, how do we factor $1 - x - x^2$? Clearly there are two factors. What do they look like? A reasonable first attempt is $1 - x - x^2 = (1 - x)(1 + x)$. But this doesn't work!

Instead, let's label the roots of $1 - x - x^2$ as α and β. (In other words, α and β are the solutions to $1 - x - x^2 = 0$.) Then

$$1 - x - x^2 = (1 - x/\alpha)(1 - x/\beta).$$

Of course we can *find* what α and β actually are using the quadratic formula, but let's not worry about this for the time being. Our goal is to rewrite the complicated fraction as a sum of two simpler fractions:

$$\frac{1}{1 - x - x^2} = \frac{}{1 - x/\alpha} + \frac{}{1 - x/\beta}.$$

What can we put in the numerators to make this work? Let's write

$$\frac{1}{1 - x - x^2} = \frac{M}{1 - x/\alpha} + \frac{N}{1 - x/\beta}.$$

Multiplying both sides by $1-x-x^2$ we find

$$1 = M(1-x/\beta) + N(1-x/\alpha).$$

This is supposed to hold for arbitrary x. Let's choose $x=\alpha$:

$$M = \frac{1}{1-\alpha/\beta} = \frac{\beta}{\beta-\alpha}$$

If instead we plug in $x=\beta$:

$$N = \frac{1}{1-\beta/\alpha} = \frac{\alpha}{\alpha-\beta}$$

In other words, we've found

$$\begin{aligned} F(x) = \frac{1}{1-x-x^2} &= \frac{\beta}{\beta-\alpha}\cdot\frac{1}{1-x/\alpha} + \frac{\alpha}{\alpha-\beta}\cdot\frac{1}{1-x/\beta} \\ &= \frac{1}{\beta-\alpha}\left(\frac{\beta}{1-x/\alpha} - \frac{\alpha}{1-x/\beta}\right) \end{aligned} \tag{$\dagger$}$$

4. THE END GAME

We started with $F(x) = \frac{1}{1-x-x^2}$, a perfectly innocent-looking function, and after some hard work converted it into something quite horrifying. Why would we do such a thing?

Our equation ($\dagger$) involved the terms

$$\frac{\beta}{1-x/\alpha} \qquad \text{and} \qquad \frac{\alpha}{1-x/\beta}.$$

Recall that α and β are some concrete numbers we could actually write down if we wanted to (they're the roots of $1-x-x^2$), so both of the above terms are secretly of the form $\frac{1}{1-r}$. This should look familiar from our very first problem set—it's the sum of a geometric series! Precisely:

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$$

Plugging this back into ($\dagger$) produces

$$\begin{aligned} F(x) &= \frac{1}{\beta-\alpha}\left(\beta + \beta\frac{x}{\alpha} + \beta\left(\frac{x}{\alpha}\right)^2 + \beta\left(\frac{x}{\alpha}\right)^3 + \beta\left(\frac{x}{\alpha}\right)^4 + \cdots - \alpha - \alpha\frac{x}{\beta} - \alpha\left(\frac{x}{\beta}\right)^2 - \alpha\left(\frac{x}{\beta}\right)^3 - \alpha\left(\frac{x}{\beta}\right)^4 - \cdots\right) \\ &= \frac{1}{\beta-\alpha}\left((\beta-\alpha) + \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)x + \left(\frac{\beta}{\alpha^2} - \frac{\alpha}{\beta^2}\right)x^2 + \left(\frac{\beta}{\alpha^3} - \frac{\alpha}{\beta^3}\right)x^3 + \cdots\right) \\ &= \frac{1}{\beta-\alpha}\left((\beta-\alpha) + \left(\frac{\beta^2-\alpha^2}{\alpha\beta}\right)x + \left(\frac{\beta^3-\alpha^3}{\alpha^2\beta^2}\right)x^2 + \left(\frac{\beta^4-\alpha^4}{\alpha^3\beta^3}\right)x^3 + \cdots\right) \end{aligned}$$

At this point we're almost done, but there's one major simplification we can make: since

$$1-x-x^2 = (1-x/\alpha)(1-x/\beta),$$

we see (using FOIL) that $\alpha\beta = -1$. Thus, continuing our argument from above,

$$\begin{aligned} F(x) &= \frac{1}{\beta-\alpha}\left((\beta-\alpha) - (\beta^2-\alpha^2)x + (\beta^3-\alpha^3)x^2 - (\beta^4-\alpha^4)x^3 + \cdots\right) \\ &= 1 - \frac{\beta^2-\alpha^2}{\beta-\alpha}x + \frac{\beta^3-\alpha^3}{\beta-\alpha}x^2 - \frac{\beta^4-\alpha^4}{\beta-\alpha}x^3 + \cdots \end{aligned}$$

On the other hand,

$$F(x) = 1 + f_1x + f_2x^2 + f_3x^3 + \cdots$$

We are therefore led to make the following bold

Conjecture 4.1. *Let α, β denote the roots of $1 - x - x^2$. Then for all positive integers n,*

$$f_n = (-1)^n \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}.$$

Our method of discovery is not totally rigorous; we summed infinite series without worrying about convergence, we assumed that if two power series are the same then all their coefficients must be the same, etc. But the point of the generating function method is to produce a conjecture—often, once you've guessed the correct result, it's not too hard to prove it (by induction, for example). We'll do this next class.

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