Friday
April 12

Last time, we did a LOT of algebra and ended on this:

Conjecture: $\forall n \in \mathbb{Z}_{>0}$,

$$f_n = (-1)^n \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha},$$

where α and β are the roots of $1-x-x^2$.

Where did this come from? Recall we started with an infinite polynomial with Fibonacci numbers as coefficients, found it ~~equal~~ was equal to $\frac{1}{1-x-x^2}$, broke that up into fractions, turned each fraction into an infinite sum, ~~and~~ combined the sums, and matched the coefficients to our initial polynomial's coefficients.

(Note: the above method is not rigorous. Does our polynomial converge? ~~Does~~ Is the matching described above valid? ~~And it~~ We'll prove it rigorously today.)

(Note: Even though we haven't solved for α and β yet, we know a lot about them. We know $1-x-x^2 = (1-\frac{x}{\alpha})(1-\frac{x}{\beta})$ $= 1 - (\frac{1}{\alpha} + \frac{1}{\beta})x + \frac{1}{\alpha\beta}x^2$. This tells us $-1 = \frac{1}{\alpha\beta} \Leftrightarrow \alpha\beta = -1$. Additionally, this tells us that $-(\frac{1}{\alpha} + \frac{1}{\beta}) = -1 \Leftrightarrow \frac{1}{\alpha} + \frac{1}{\beta} = 1 \Leftrightarrow \frac{\alpha+\beta}{\alpha\beta} = 1$ $\Leftrightarrow \alpha + \beta = \alpha\beta$. $\alpha\beta = -1$, so $\alpha+\beta = -1$.)

Let's now prove the conjecture.

Proof: By strong induction.

Let $A := \{n \in \mathbb{Z}_{>0} : f_n = (-1)^n \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\}$.

(We're going to start by proving the inductive step and come back to the base cases later.)

Suppose $k \in A \ \forall k < n$. Want: $n \in A$.

In particular, we know $n-1 \in A$ and $n-2 \in A$.

Since $f_n = f_{n-1} + f_{n-2}$ (by the structure of the Fibonacci numbers), we have $f_n = f_{n-1} + f_{n-2} = (-1)^{n-1} \frac{\beta^n - \alpha^n}{\beta - \alpha} + (-1)^{n-2} \frac{\beta^{n-1} - \alpha^{n-1}}{\beta - \alpha}$
$= \frac{(-1)^n}{\beta - \alpha}\left(-(\beta^n - \alpha^n) + (\beta^{n-1} - \alpha^{n-1})\right) = \frac{(-1)^n}{\beta - \alpha}\left((\beta^{n-1} - \beta^n) - (\alpha^{n-1} - \alpha^n)\right)$
$= \frac{(-1)^n}{\beta - \alpha}\left(\beta^{n-1}(1-\beta) - \alpha^{n-1}(1-\alpha)\right)$.

Now, recall that α and β are roots of $1 - x - x^2$. This means $1 - \alpha - \alpha^2 = 0$ and $1 - \beta - \beta^2 = 0$, meaning $1 - \alpha = \alpha^2$ and $1 - \beta = \beta^2$. Therefore,
$\frac{(-1)^n}{\beta - \alpha}\left(\beta^{n-1}(1-\beta) - \alpha^{n-1}(1-\alpha)\right) = \frac{(-1)^n}{\beta - \alpha}\left(\beta^{n-1} \cdot \beta^2 - \alpha^{n-1} \cdot \alpha^2\right) =$
$\frac{(-1)^n}{\beta - \alpha}\left(\beta^{n+1} - \alpha^{n+1}\right) = (-1)^n \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}$. Therefore, $n \in A$.

Now it's time for the base cases! First, note that to prove $n \in A$, we needed to use $n-1 \in A$ and $n-2 \in A$, so we need two base cases. (In terms of dominos, we need to knock down the first two dominos to ensure the third falls.)

Base cases:

$1 \in A$: $(-1)^1 \cdot \frac{\beta^2 - \alpha^2}{\beta - \alpha} = (-1)\frac{(\beta+\alpha)(\beta-\alpha)}{\beta - \alpha} = -(\beta + \alpha)$. In an earlier ~~one~~ note, we showed $\alpha + \beta = -1$, so $-(\beta + \alpha) = 1$. ~~Thus, $1 \in A$~~ ~~$2 \in A$~~ Also, $f_1 = 1$. $1 = 1$, so $1 \in A$.

$2 \in A$: $(-1)^2 \cdot \frac{\beta^3 - \alpha^3}{\beta - \alpha} = \frac{(\beta - \alpha)(\beta^2 + \beta\alpha + \alpha^2)}{\beta - \alpha} = \beta^2 + \beta\alpha + \alpha^2 = (\alpha + \beta)^2 - \alpha\beta$
$= (-1)^2 - (-1) = 2$. Also, $f_2 = 2$. $2 = 2$, so $2 \in A$. $\blacksquare$

(Hey, what's that box? Well, drawing a little box is a touch quicker than writing "Q.E.D.," but they're synonymous.)

(Note: math is useful! We used $\beta^2 - \alpha^2 = (\beta + \alpha)(\beta - \alpha)$ in the proof, but $x^2 - 1 = (x-1)(x+1)$ is useful in other instances—for example, we can compute $14 \cdot 16 = (15-1)(15+1) = 15^2 - 1 = 224$ relatively quickly.)

Now, recall this conjecture from Kepler.

Conjecture (Kepler):

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$$

Proof: $\displaystyle\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{(-1)^{n+1}\frac{\beta^{n+2}-\alpha^{n+2}}{\beta-\alpha}}{(-1)^{n}\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}} = -\lim_{n\to\infty} \frac{\beta^{n+2}-\alpha^{n+2}}{\beta^{n+1}-\alpha^{n+1}}.$

Well, we're out of luck. We must compute α and β to proceed. Let's do it!

$1-x-x^2=0 \Leftrightarrow x^2+x-1=0 \Rightarrow \alpha, \beta = \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}.$

Let's say $\beta = \frac{-1+\sqrt{5}}{2}$ and $\alpha = \frac{-1-\sqrt{5}}{2}$.

Note that $2<\sqrt{5}<3 \Rightarrow 1<-1+\sqrt{5}<2 \Rightarrow \frac{1}{2}<\beta<1$.

This is great! This means as $n\to\infty$, β^{n+2} and β^{n+1} go to 0. Thus, $\displaystyle\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = -\lim_{n\to\infty}\frac{\beta^{n+2}-\alpha^{n+2}}{\beta^{n+1}-\alpha^{n+1}} = -\lim_{n\to\infty}\frac{-\alpha^{n+2}}{-\alpha^{n+1}} = -\lim_{n\to\infty}\alpha$ $= -\alpha = \frac{1+\sqrt{5}}{2}$. $\blacksquare$

Final note (that will be explored on the problem set):
Zeckendorf's Theorem (discovered by Lekkerkerker in the 1950s):

Every $n\in\mathbb{Z}$ can be expressed as the sum of distinct, non-consecutive Fibonacci numbers in a unique way (e.g. $25 = 21+3+1 = f_7+f_3+f_1$).