Wednesday, April 17

Let's address some homework stuff before we begin. Are we allowed to use f_0 in Zeckendorf decomposition? No! Why not? Well, the decomposition is supposed to be unique. If we allow f_0, $3=f_3=f_2+f_0$, so the decomposition is not unique.

Fun note: the ratio between consecutive ~~...~~ Fibonacci numbers is approximately the ratio between miles and kilometers, so we can convert from miles to kilometers by moving to the next Fibonacci number (for instance, 55 mph $\approx$ 89 km/h).

What if the number of miles per hour isn't a Fibonacci number? ZECKENDORF DECOMPOSITION! Just do the shift with the ~~...~~ Fibonacci numbers that comprise the number (for instance, 65 mph = 55+8+2 mph $\approx$ 89+13+3 km/h = 105 km/h).

One problem on the homework requires a uniqueness proof. Let's do an example with binary.

Theorem: Every positive integer has a unique binary expansion.

Idea: $75=64+8+2+1=2^6+2^3+2^1+2^0$.

How did we get this? We took the largest power of 2 less than 75, then the largest power of 2 less than 75−64, and so on. (This is called a greedy algorithm.)

If 64 weren't in our expansion, could we reach 75? No! Why not? If we added $2^7=128$, we'd overshoot the goal, and $2^5+2^4+2^3+2^2+2^1+2^0=32+16+8+4+2+1=63<75$, so we need 64.

Now let's prove it!

Proof of theorem:

Let m be the least positive integer that has multiple binary expansions, say $m = 2^k$ + smaller powers of $2 = 2^\ell$ + smaller powers of 2.

Without loss of generality, say $k \geq \ell$.

If $k \neq \ell$ (i.e. $k > \ell$), then

$m = 2^\ell$ + smaller powers of $2 \leq 2^\ell + 2^{\ell-1} + 2^{\ell-2} + \ldots + 2 + 1 = 2^{\ell+1} - 1$

$\leq 2^k - 1 < m$. This is a contradiction, so $k = \ell$.

Since m is minimal, $m - 2^k$ has a unique binary expansion.

Thus, m has a unique binary expansion. $\blacksquare$

We're going to slightly modify notation from last time.

We wrote $82 \equiv 5 \pmod 7$, but for now, we'll write

$82 \bmod 7 = 5$. (This means $82 \div 7$ has a remainder of 5, or, equivalently, $82 = 7 \times 11 + 5$.)

Question: What time will it be 80 hours from now?

Section 1: It's 10:00(ish). In 80 hours, it'll be 90:00(ish), which doesn't make sense. Clocks work mod 12, and $90 \bmod 12 = 6$, so in 80 hours, it'll be 6:00(ish).

Section 2: It's 11:00(ish). In 80 hours, it'll be 91:00(ish), which doesn't make sense. Clocks work mod 12, and $91 \bmod 12 = 7$, so in 80 hours, it'll be 7:00(ish).

(Something to think about: Are those times am or pm? How could we figure that out?)

Definition: For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$, we can write $a = qn + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < n$. Then $a \pmod n = r$.

We showed this $a = qn + r$ business on the homework, but we didn't show that q and r were unique. What if there's some x such that $x \pmod 7 = 3$ and $x \pmod 7 = 5$? Let's prove some stuff.

Theorem (quotient remainder theorem):

Given $a \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$, $\exists! q, r \in \mathbb{Z}$ s.t. $a = qn + r$ and $0 \leq r < n$.

[Note: $\exists!$ means "there exists a unique.")

Existence of q and r:

Let $q := \lfloor \frac{a}{n} \rfloor$. Then $q \leq \frac{a}{n} < q+1 \Rightarrow qn \leq a < qn+n \Rightarrow 0 \leq a - qn < n$. Let $r := a - qn$. Thus q and r exist.

Uniqueness of q and r:

Suppose $a = qn + r = q'n + r'$ for some $q, q', r, r' \in \mathbb{Z}$ with $0 \leq r, r' < n$. Then $qn - q'n = r' - r \Rightarrow |(q-q')n| = |r' - r| \leq n-1 \Rightarrow |q - q'| \leq \frac{n-1}{n} < 1 \Rightarrow q - q' = 0 \Rightarrow q = q' \Rightarrow 0 = r' - r \Rightarrow r = r'$. ∎