Monday, April 29

We're moving on to combinatorics!
But first, what is $(x+y)^2$? It's $x^2+2xy+y^2$.
Alright, what about $(x+y)^3$? That's $x^3+3x^2y+3xy^2+y^3$.
Let's do one more... $(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$.
Okay, that's all well and good, but what's the point?
Notice:

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}$$

There's a really nice pattern showing up with the coefficients.
Let's isolate those!

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\vdots
\end{array}$$

This is called Pascal's triangle.
(Fun side note: in China, it's sometimes called Yang Hui's triangle, and in Iran, it's sometimes called Khayyam's triangle. However, it was discussed in France before Pascal, in China before Yang Hui, and in Iran before Khayyam!)

There are A LOT of patterns here. What are some?

① Each entry is the sum of the entries above left and above right.

② Each row is a palindrome.

③ In row p, every entry (excluding the 1s at the ends) is congruent to $0 \pmod{p}$.

For the next few patterns, it'll help to redraw it slightly.
~~This is the "Right~~ We'll call this "Right Pascal's Triangle":

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
⋮

④ In Right Pascal's Triangle, the second column is the counting numbers and the third column is ~~the~~ the triangular numbers.

⑤ In Right Pascal's Triangle, the diagonals (drawn above in red) sum to Fibonacci numbers.

⑥ The sum of the n^{th} row is 2^n.

We can state things all day long, but why are these patterns true? Let's try to prove some of these.

Proofs...?

① ? Why is the recursion there?

(2) x and y are symmetric (we could swap them in $(x+y)^n$).
(3)? Why is this true?
(4) This follows from recursion (but why is recursion true?).
(5)? Why is this true?
(6). If we let $x=1$ and $y=1$, $(x+y)^n$ will just be the coefficients, and $(x+y)^n = 2^n$.

IS THERE A NICE FORMULA FOR THE COEFFICIENTS?

First, a warm-up:

$(a_1+a_2)(b_1+b_2+b_3) = a_1b_1 + a_1b_2 + a_1b_3 + a_2b_1 + a_2b_2 + a_2b_3.$

Alright, then what's $(a_1+a_2)(b_1+b_2+b_3)(c_1+c_2)$?

I don't want to write out all those terms! Let's describe it. This is the sum of all terms consisting of one of the a's times one of the b's times one of the c's.

With this in mind, what is $(x+y)^{17}$?

We'll come back to this later, but the coefficient on $x^k y^{17-k}$ is the number of ways of choosing k x's out of 17 x's.