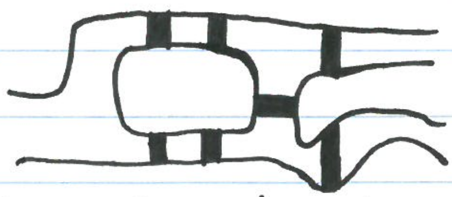
Friday,
May 3

Let's talk about Königsberg. (It used to be in Prussia, but now it's in Russia, and it's called Kaliningrad.)
Question: Is it possible to take a walk around Königsberg, crossing each of its bridges exactly once?
Let's draw the map:



After a few attempts, we couldn't find anything. Is it impossible?

Euler proved that no such path exists.
(Side note: Euler was awesome.)
His first idea was to abstract the problem. Let's draw a vertex representing each landmass and connect each vertex by a bridge (or however many bridges there are). This yields:

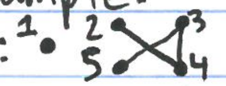
Problem: If we start at some vertex α and end at some vertex ω, is it possible to walk along the bridges, crossing each exactly once?
A typical walk looks like:
$\alpha \rightsquigarrow w \rightsquigarrow z \rightsquigarrow t \rightsquigarrow \dots \rightsquigarrow \omega$.
Pick any vertex v that isn't α or ω. Our walk looks like:
$\alpha \sim \dots \sim v \sim \dots \sim v \sim \dots \sim \omega$.
Can v have an odd number of ~~brige~~ bridges coming out of it? No! If it did, after ~~entering~~ going to and from v, we'd be guaranteed to have a bridge left over or to reuse a bridge, so we wouldn't use each exactly once!

Thus, v must have an even number of bridges coming out of it. In the Königsberg example, though, each vertex has an odd number of bridges! Thus, there cannot exist a walk crossing each bridge exactly once. Hooray!

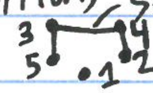
This motivates:

Definition: A graph is a collection of vertices, V, and a collection of edges, E, connecting pairs of vertices such that no edge connects a vertex to itself.

Non-example: 

Example: 

(this has five vertices and three edges). Our earlier Königsberg ~~gra~~ abstraction is a graph with 4 vertices and 7 edges.

(Note that vertices don't have fixed positions, so the above numbered example is the same as 

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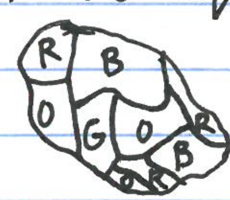
Problem: Given a map of a bunch of countries, we want to color in the map in such a way that any two countries that share a border are different colors. How many colors are required?

Example: 

By trying to color this in, we found we needed four colors.

Theorem: Every map can be colored using 4 colors (aptly named the Four Color Theorem).

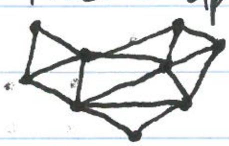
(The proof of this requires a computer. We'll prove it for six colors later.)

The proof uses graph theory. Let's turn the map into a graph! Assign a vertex to each country. Connect a pair of ~~vertices~~ countries with an edge if and only if they share a border.

Now the question becomes:

Q: How few colors do we need such that no edge has both endpoints the same color?

(Going back to turning the map into a graph for a moment, the map we had earlier as a graph is:



That's cool!)

We can also model concepts using graph theory.

Example: We want to schedule final exams in such a way that no student is scheduled to take two exams simultaneously. Make each class a vertex. Draw an edge between two vertices if and only if there's a student in both classes.

Thus, the problem becomes coloring each vertex with a time slot, using as few time slots as possible such that no edge has both endpoints the same color.

Wait a second... THAT'S JUST THE MAP PROBLEM!