Wednesday, May 8

We're still talking about graphs! Let's talk about notation:

V = set of all vertices

V = number of vertices

E = set of all edges

E = number of edges

$\deg(v)$ = number of edges coming out of v

Connected component: for any two vertices v, w, there exists a path from v to w.

Last time, we discovered and proved:

Proposition: $$\sum_{v \in V} \deg(v) = 2E.$$

Let's take both sides (mod 2):

$$\sum_{v \in V} \deg(v) \equiv 0 \pmod 2.$$

We can split up ~~the~~ the sum, since each degree is even or odd:

$$\sum_{\substack{v \in V \\ \deg(v) \text{ is even}}} \deg(v) + \sum_{\substack{v \in V \\ \deg(v) \text{ is odd}}} \deg(v) \equiv 0 \pmod 2.$$

If $\deg(v)$ is even, $\deg(v) \equiv 0 \pmod 2$. If $\deg(v)$ is odd, $\deg(v) \equiv 1 \pmod 2$. Therefore,

$$\sum_{\substack{v \in V \\ \deg(v) \text{ is odd}}} 1 \equiv 0 \pmod 2.$$

The left hand side is just counting the number of vertices of odd degree, though! For it to be $0 \pmod 2$, there must be an even number of vertices of odd degree.

This proves:

Proposition (Handshake Lemma): In any graph, the number of vertices of odd degree is even.

We're moving on!

Efficient Connectivity

We want to find the "skeleton" of a graph: the smallest subgraph that contains all vertices of the original and is connected.

Example:

 We can remove edges and still have a connected graph: 

We can still remove an edge: 

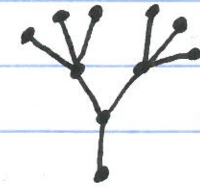
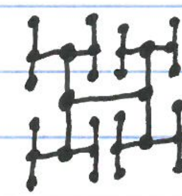
Removing any edge disconnects this, so we're done.

This is called a minimal spanning tree of a graph.

Minimal spanning trees are not unique. Using the above example, we could have ended with  or .

Minimal spanning trees are examples of...

Definition: A graph is called a tree if and only if there exists precisely one path (that doesn't repeat edges).

Examples:  

Proposition: G is a tree if and only if it is connected and acyclic (meaning there are no cycles (meaning an Eulerian path starting and ending at the same vertex)).

We are not going to prove this, but the proof isn't too bad.

Example: The Königsberg graph is not acyclic:

(There's a cycle between the top left and middle left vertices, for instance.)

AWESOME TREE OBSERVATION: In a tree, there's a relationship between the number of vertices and number of edges: $V = E + 1$.

(We're not going to prove this, but it makes sense that adding an edge to a tree introduces a cycle.)

This is not true in general:  has $V=4$ and $E=5$, for example.

Question: Is there a relationship that does hold for any graph?

Note: Vertices are 0-dimensional, edges are 1-dimensional. What's the 2-dimensional analogue? Faces!

"Definition" (we'll see later that it's flawed): A face is an enclosed connected 2-dimensional space in a graph.

~~"Examples"~~ "Examples":

			
$V=4$ $E=2$ $F=0$	$V=4$ $E=3$ $F=1$	Wait a second... there's no vertex in the middle! This is:	$V=4$ $E=3$ $F=2$

Because of that third example, we restrict attention to planar graphs: graphs that can be represented without intersecting edges.

There's a better way to count faces: <u>count the number of distinct 2-dimensional pieces the graph splits the plane into.</u>

This counts the "outside region", so the number of faces in the above examples needs to be bumped up 1.

Euler proved:

Euler's Theorem: In any connected planar graph, $V-E+F=2$. (This also works for polyhedra!)

Corollary (next time): In any simple (i.e. no multiple edges), planar graph, there is a vertex with degree less than or equal to 5.