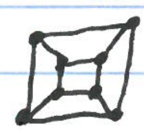
Friday,
May 10

This is the final class period!
Last time, we discussed Euler's theorem:

$$\underset{\text{0 dimensional}}{V} - \underset{\text{1 dimensional}}{E} + \underset{\text{2 dimensional}}{F} = 2$$

for any connected planar graph.
This doesn't only work for planar graphs. $V-E+F$ is called the Euler characteristic, and for differing classes of geometric objects, we'll get different constants.
What if we use a polyhedron? For example, a tetrahedron has 4 vertices, 6 edges, and 4 faces, and $4-6-4=2$. A cube has 8 vertices, 12 edges, and 6 faces, and $8-12+6=2$.
Wait a second! Polyhedra aren't planar graphs...or are they?
Consider the shadow of a tetrahedron and a cube:

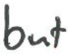
 

THESE ARE PLANAR GRAPHS! It makes sense, then, that $V-E+F=2$ for polyhedra.
(Something to ponder... while talking about planar graphs, we had to include the outer face. There is no outer face on a polyhedron. What happens?)
We will not prove Euler's theorem, and we don't have to know the proof for the exam. However, there will be two provided. One is in section 53 of the text. The other is from Red Burton & Stephanie Mathews.
It will be posted on the course website.

(Note: Red Burton is a computer program, not a person.)

Applization of Euler's Theorem

Given any connected, simple, planar graph, $\exists v \in V$ such that $\deg(v) \le 5$.

(Note: "Simple" means that there aren't multiple edges connecting two vertizes, so a situation like  is not allowed, but  is.)

Proof| We will prove this by contradiction. Suppose there is no such vertex, i.e. $\deg(v) \ge 6$ $\forall v \in V$.

Step 1: Relate V and E.

We showed before that $\sum_{v \in V} \deg(v) = 2E$.

Since $\deg(v) \ge 6$ for every v, we have:

$$6V = \sum_{v \in V} 6 \le \sum_{v \in V} \deg(v) = 2E \Rightarrow 3V \le E.$$

Step 2: Relate F and E.

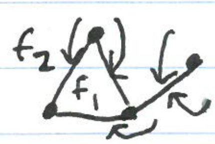
It turns out,

$$\sum_{f \in F} \deg(f) = 2E,$$

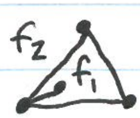
where $\deg(f)$ = the number of sides of edges attached to face f. (We're not going to prove this.)

Examples:

$\deg(f_1) = 3$
$\deg(f_2) = 3$



$\deg(f_1) = 3$
$\deg(f_2) = 5$



$\deg(f_1) = 5$
$\deg(f_2) = 3$

Note: $\deg(f) \ge 3$ except possibly for the external face, which can have $\deg(f) = 2$. Thus,

$$3F-1 = 2 + \sum_{\substack{f\in F \\ \text{minus} \\ \text{exterior} \\ \text{face}}} 3 \le \sum_{f\in F} \deg(f) = 2E \Rightarrow 3F-1 \le 2E.$$

Step 3: Combine and win!

From steps 1 and 2, adding the inequalities yields

$$3F-1+3V \le 3E$$
$$\Rightarrow 3V-3E+3F \le 1$$
$$\Rightarrow 3(V-E+F) \le 1$$
$$\Rightarrow 3\cdot 2 \le 1 \Rightarrow 6 \le 1.$$ This is codswallop! ∎

We'll end on the idea of how to prove the six color theorem.

Corollary: Any simple connected planar graph can be 6-colored.

Proof idea: By strong induction on V.

Given a graph G, there exists v_0 such that $\deg(v_0) \le 5$.

Remove v_0 and delete its edges.

Each connected piece of what's left has fewer vertices than G. Hence, by strong induction, they can be 6-colored. Now reattach v_0 and color it with a color none of its less than or equal to 5 neighbors has been colored with. Done! ∎

Have a great summer, everyone!