Instructor: Leo Goldmakher

NAME (LAST, NICKNAME): ______________________

SECTION # (10AM = 1, 11AM = 2): ____________

Williams College
Department of Mathematics and Statistics

MATH 200 : DISCRETE MATH

MIDTERM – due Wednesday, March 13th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp.
Midterms submitted later than this will not be graded.

For the purposes of this exam, you **may** use: the textbook for the course, any graded work you've gotten back, the course website and anything posted on there. You may **not** use: any calculators, any books other than the official textbook, any online resources apart from the course website. You may ask me questions during the course of the exam either over email or in person, but you may not interact with any other person about the exam until Thursday, March 14th.

Please print and attach this page as the first page of your submitted midterm.

PROBLEM	GRADE
M.1	
M.2	
M.3	
M.4	
M.5	
M.6	
M.7	
Total	

Please print this page and sign the following statement **before starting the midterm:**

I understand that I am not allowed to use the internet to assist with this exam, apart from accessing the course website or emailing the instructor. I also understand that I cannot request assistance on (nor discuss content from) this exam with anyone other than the instructor. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________________

Midterm

M.1 Consider the following predicate P: *Between any two positive perfect squares there's always a prime.* Express $\neg P$ using quantifier and relation symbols, without using 'not' anywhere in your statement (implicitly or explicitly). [*Aside: empirically P seems to be true, but no one has been able to prove it.*]

M.2 The goal of this problem is to give an alternative proof that $\sqrt{2}$ is irrational (so for this problem you should forget about the proof we did in class!). Let $\mathcal{A} := \{n \in \mathbb{Z}_{>0} : n\sqrt{2} \in \mathbb{Z}\}$.

(a) Prove that if $k \in \mathcal{A}$, then $(\sqrt{2} - 1)k \in \mathcal{A}$.

(b) Use part (a) to show that the set $\mathcal{A}$ must be empty. [*Hint: warm up by using (a) to prove that* $1 \notin \mathcal{A}$.]

(c) Use part (b) to prove that $\sqrt{2} \notin \mathbb{Q}$. [*Your answer should be very short!*]

M.3 We define a new logical connective as follows:

$$P \uparrow Q := \neg(P \wedge Q).$$

This connective is cool because it can be used to replace all the other standard ones, as you'll see below.

(a) The proposition $P \uparrow P$ is logically equivalent to a very familiar expression. Which one?

(b) Express $\implies$ in terms of $\uparrow$. (In other words, write down an expression that's purely in terms of P's, Q's, and $\uparrow$'s that is logically equivalent to $P \implies Q$.) Prove the logical equivalence with a truth table.

(c) Express $\wedge$ in terms of $\uparrow$, and prove with a truth table.

(d) Express $\vee$ in terms of $\uparrow$, and prove with a truth table.

(e) Rewrite the proposition $P \wedge \big((\neg Q) \implies (P \vee Q)\big)$ purely in terms of P's, Q's, and $\uparrow$'s. You don't need a truth table for this one. [*Hint: Use (a)-(d) to make your life easier! The main take-away from this part is that* any *boolean expression can be expressed in terms of just* $\uparrow$.]

(f) Find a formula expressing $\#(P \uparrow Q)$ in terms of $\#P$ and $\#Q$.

M.4 A *perfect cube* is any real number of the form a^3 for some $a \in \mathbb{Z}$. (For example, 0, -8, and 125 are all perfect cubes, whereas 4 is not a perfect cube.) Prove that $n^3 + 2$ is a perfect cube if and only if $n = -1$. [*Hint: recall from problem* **1.3** *that* $1 + x + x^2 = \frac{1-x^3}{1-x}$. *Don't feel compelled to use this if it's not helpful!*]

M.5 Let $\mathcal{A}$ denote the set of all real numbers in $(0, 1)$ that can be expressed as a finite decimal. Thus, for example, $\frac{1}{3} \notin \mathcal{A}$ (because $\frac{1}{3} = 0.3333...$), whereas $\frac{1}{4} \in \mathcal{A}$ (because $\frac{1}{4} = 0.25$). Is $\mathcal{A}$ countable? If yes, prove with an enumeration of the set. If no, prove why not.

M.6 Suppose a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies two nice properties:

- $f(1) \neq 0$, and
- $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathbb{Z}$.

(a) Prove that $f(n) = 0$ if and only if $n = 0$.

(b) Prove that $f(-n) = -f(n)$ for every $n \in \mathbb{Z}$.

(c) Prove that f must be injective.

(d) Is f surjective? Prove your answer.

M.7 Prove that $(0, 1] \approx (0, 1)$ by giving an explicit function producing a one-to-one correspondence.
(Here $(0, 1]$ means the set $\{x \in \mathbb{R} : 0 < x \leq 1\}$.)