Instructor: Leo Goldmakher

Name (Last, Nickname): ____________________

Section # (10am = 1, 11am = 2): ____________

Williams College
Department of Mathematics and Statistics

MATH 200 : DISCRETE MATH

Problem Set 6 – due Thursday, April 4th

INSTRUCTIONS:

This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Late assignments may be submitted at the beginning of Friday's class to me in person (i.e. don't leave them in my mailbox or ask someone else to submit on your behalf); however, 5% will be deducted for late submission. *Assignments submitted later than start of class on Friday will not be graded.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
6.1	
6.2	
6.3	
6.4	
6.5	
6.6	
Total	

Please read the following statement and sign **before writing the final version of this problem set:**

I understand that I am not allowed to use the internet to assist with this assignment, apart from accessing the course website or looking up definitions. I also understand that I must write down the final version of my assignment in isolation from any other person, and to not copy from any set of written notes created when another person was present. I pledge to abide by the Williams honor code.

SIGNATURE:____________________________________

Problem Set 6

6.1 Given a function $f : A \to B$, we define the function $f^{-1} : B \to \mathcal{P}(A)$ (called the *inverse of* f) as follows:

$$f^{-1}(\beta) := \{\alpha \in A : f(\alpha) = \beta\}.$$

Note that f^{-1} is mapping B to the *power set* of A; its outputs are subsets of A.

(a) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $x \mapsto x^2$. Find $f^{-1}(0)$, $f^{-1}(2)$, and $f^{-1}(4)$.

(b) Carefully explain the assertion: if $f : A \hookrightarrow\!\!\!\!\rightarrow B$, we may interpret f^{-1} as a function from B to A.

(c) Show (by example) that it's possible for the interpretation from (b) to fail to be a function from B to A for an injection $f : A \hookrightarrow B$.

6.2 I pick $a, b \in \mathbb{Z}$ with $b \neq 0$.

(a) Prove that $(a - b\mathbb{Z}) \cap \mathbb{Z}_{>0} \neq \emptyset$. [Recall that $a - b\mathbb{Z} := \{a - bn : n \in \mathbb{Z}\}$.]

(b) Prove that there exist $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < |b|$. [*Hint: use part* (a).]

(c) Compute q and r given $a = 1003$ and $b = 9$. No calculators allowed!

(d) Use (b) to prove that every integer must be either even or odd.

(e) Use part (d) and the pigeonhole principle to prove that in any set of three integers, two of them have even difference.

(f) Fix a positive integer n. Prove that in any set of $n + 1$ integers, two of them have difference that's a multiple of n. [*Hint: use parts* (d) *and* (e) *as inspiration.*]

6.3 I choose 51 distinct numbers from the set $\{1, 2, 3, \ldots, 100\}$. Prove that one of my numbers must be a multiple of another one of my numbers. [*Hint: every positive integer can be written in the form* $2^n \times$ *odd.*]

6.4 Color each point in the plane red, green, or blue. Prove that there must exist two points of the same color that are exactly one unit apart. [*Aside: how many colors are required to guarantee that no two points that are one unit apart have the same color? This is a famous open problem, with a fascinating recent breakthrough by Aubrey de Grey. Check out* `https://goo.gl/XzXsST` *for more info.*]

6.5 What's wrong with the following proof? Identify the issue(s) as precisely as possible.

Claim. *All English words have the same number of letters.*

'*Proof.*' We will prove (by induction) that in any finite collection of English words, all the words in the collection have the same number of letters. Since there are finitely many words in the English language, we deduce that all words must have the same number of letters.

Let

$$\mathcal{A} := \{n \in \mathbb{Z}_{>0} : \text{in every set of } n \text{ English words, all the words have the same number of letters}\}.$$

Clearly $1 \in \mathcal{A}$, since in any set consisting of a single word, all the words in that set have the same number of letters.

Now suppose $k \in \mathcal{A}$. Consider any set S consisting of $k+1$ words, and label these words $a_1, a_2, \ldots, a_{k+1}$. All the words $a_1, a_2, \ldots, a_k$ have the same number of letters, since this is a set of k words and $k \in \mathcal{A}$; note in particular that a_1 and a_2 have the same number of letters. But also, the words $a_2, \ldots, a_{k+1}$ all have the same number of letters, again since $k \in \mathcal{A}$; note in particular that a_2 and a_{k+1} have the same number of letters. Thus both a_1 and a_{k+1} have the same number of letters as a_2, whence all the words in S must have the same number of letters. Since S was an arbitrary set of $k + 1$ English words, we've proved that $k + 1 \in \mathcal{A}$. By induction, $\mathcal{A} = \mathbb{Z}_{>0}$, which proves the claim. 'QED'

6.6 Induction practice!

(a) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

(b) Prove that for all positive integers n,

$$(1)(2)(3) + (2)(3)(4) + (3)(4)(5) + ... + (n)(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

(c) Conjecture a formula for the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$, and prove it using induction.

(d) Prove that for any positive integer n, the number $1^3 + 2^3 + 3^3 + \cdots + n^3$ is a perfect square.