Instructor: Leo Goldmakher

Name (Last, Nickname): ____________________

Section # (10am = 1, 11am = 2): ____________

Williams College
Department of Mathematics and Statistics

MATH 200 : DISCRETE MATH

Problem Set 7 – due Thursday, April 11th

INSTRUCTIONS:

This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Late assignments may be submitted at the beginning of Friday's class to me in person (i.e. don't leave them in my mailbox or ask someone else to submit on your behalf); however, 5% will be deducted for late submission. *Assignments submitted later than start of class on Friday will not be graded.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
7.1	
7.2	
7.3	
7.4	
7.5	
7.6	
7.7	
Total	

Please read the following statement and sign **before writing the final version of this problem set:**

I understand that I am not allowed to use the internet to assist with this assignment, apart from accessing the course website or looking up definitions. I also understand that I must write down the final version of my assignment in isolation from any other person, and to not copy from any set of written notes created when another person was present. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 7

7.1 Use strong induction to prove that in any set of five consecutive positive integers, precisely one of them is divisible by 5.

7.2 Use strong induction to prove that every positive integer can be written as a product of primes. (Here you can interpret the number 1 as the 'empty' product, consisting of no primes multiplied together.)

7.3 Discover and prove a formula for

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$$

7.4 Suppose a set $\mathcal{A} \subseteq \mathbb{Z}_{>0}$ satisfies two properties:

(i) $13 \in \mathcal{A}$, and

(ii) $n \in \mathcal{A} \implies n+1 \in \mathcal{A}$.

Prove that $\{n \in \mathbb{Z} : n \geq 13\} \subseteq \mathcal{A}$. [*Hint: use induction!*]

7.5 The goal of this problem is to give another proof that $\sqrt{2} \notin \mathbb{Q}$, discovered by John Conway. Let

$$\mathcal{A} := \{n \in \mathbb{Z}_{>0} : n\sqrt{2} \in \mathbb{Z}\}.$$

We'll also need the concept of the *fractional part of* α, defined

$$\langle \alpha \rangle := \alpha - \lfloor \alpha \rfloor.$$

For example, $\langle 2.34 \rangle = 0.34$, and $\langle 3 \rangle = 0$.

(a) Prove that if $a, b \in \mathbb{Z}_{>0}$ and $\frac{a}{b} \geq 1$, then there exists $b' \in \mathbb{Z}$ such that $0 \leq b' < b$ and $\left\langle \frac{a}{b} \right\rangle = \frac{b'}{b}$.

(b) Suppose $n \in \mathcal{A}$. Prove that there must exist $k \in \mathbb{Z}$ such that $\sqrt{2} = \frac{k}{n} = \frac{2n}{k}$.

(c) Keeping the notation as above, prove that $\exists k', n' \in \mathbb{Z}$ such that $0 < k' < k$, $0 < n' < n$, and $\frac{n'}{n} = \frac{k'}{k}$.

(d) Keeping the notation as above, prove that $n' \in \mathcal{A}$, and deduce that $\mathcal{A} = \emptyset$.

7.6 A well-known puzzle game works as follows. (See figure below.) There are three vertical posts, numbered 1, 2, and 3. At the start of the game, there's a stack of n concentric rings of decreasing radius stacked on post 1, and the other two posts are empty. The goal is to end up with the entire stack of rings on post 3, again in order of decreasing radius. A move consists of transporting a ring at the top of any stack to another post; however, no ring can be placed on top of a smaller ring. Prove that the entire stack of n rings can be moved onto post 3 in $2^n - 1$ moves, and that this cannot be done in fewer than $2^n - 1$ moves.

7.7 Use strong induction to prove the well-ordering principle. [*Hint: Suppose $\mathcal{A} \subseteq \mathbb{Z}_{>0}$ has no least element, and consider the complement of $\mathcal{A}$ (i.e. the set $\mathbb{Z}_{>0} \setminus \mathcal{A}$).*]

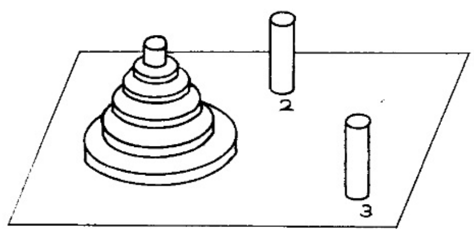

Figure 1: Initial set-up of game in Problem **7.6**