Instructor: Leo Goldmakher

NAME (LAST, NICKNAME): ____________________

SECTION # (10AM = 1, 11AM = 2): ____________

Williams College
Department of Mathematics and Statistics

MATH 200 : DISCRETE MATH

Problem Set 8 – due Thursday, April 18th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Late assignments may be submitted at the beginning of Friday's class to me in person (i.e. don't leave them in my mailbox or ask someone else to submit on your behalf); however, 5% will be deducted for late submission. *Assignments submitted later than start of class on Friday will not be graded.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
8.1	
8.2	
8.3	
8.4	
8.5	
Total	

Please read the following statement and sign **before writing the final version of this problem set:**

I understand that I am not allowed to use the internet to assist with this assignment, apart from accessing the course website or looking up definitions. I also understand that I must write down the final version of my assignment in isolation from any other person, and to not copy from any set of written notes created when another person was present. I pledge to abide by the Williams honor code.

SIGNATURE:____________________________________

Problem Set 8

Throughout this problem set, let f_n denote the n^{th} Fibonacci number.

8.1 Discover and prove a simple formula for the quantity $f_n^2 - f_{n-1}f_{n+1}$ that holds for all $n \geq 2$.

8.2 Consider the sequence

$$\underbrace{1}_{x_1}, \underbrace{1+\frac{1}{1}}_{x_2}, \underbrace{1+\cfrac{1}{1+\cfrac{1}{1}}}_{x_3}, \underbrace{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}_{x_4}, \underbrace{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}_{x_5}, \ldots$$

(a) Conjecture and prove a formula for x_n that is in the form of a simple fraction (i.e. in the form $\frac{a}{b}$).

(b) Use part (a) to prove Kepler's conjecture that the ratio between consecutive Fibonacci numbers tends to the 'golden ratio' $\frac{1+\sqrt{5}}{2}$. [*Hint: set*

$$\alpha := 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}},$$

where the fraction continues infinitely downwards, and write an equation expressing α in terms of itself.]

8.3 Given a positive integer n, consider the sum

$$S(n) := f_{n-1} + f_{n-3} + f_{n-5} + \cdots,$$

where the sum runs until the smallest positive index possible; formally we could express this sum as

$$S(n) := \sum_{0 \leq k < \frac{n-1}{2}} f_{n-(2k+1)}.$$

For example, $S(6) = f_5 + f_3 + f_1 = 12$ and $S(7) = f_6 + f_4 + f_2 = 20$. Prove that $S(n) < f_n$ for all $n \geq 2$.

8.4 The goal of this problem is to explore the Zeckendorf decomposition discussed in class.

(a) Prove that any $n \in \mathbb{Z}_{>0}$ can be expressed as the sum of distinct non-consecutive Fibonacci numbers. (For example, $12 = f_5 + f_3 + f_1$ and $13 = f_6$.) [*Hint: use strong induction. Given $k \geq 3$, there exists a unique n such that $f_n \leq k < f_{n+1}$. Thus...*]

(b) Suppose $\mathcal{A}$ is a set of non-consecutive Fibonacci numbers all of which are $< f_n$. Prove that the sum of all the elements in $\mathcal{A}$ is $< f_n$. [*Hint: do problem **8.3** first.*]

(c) Prove that any $n \in \mathbb{Z}_{>0}$ can be expressed as the sum of distinct non-consecutive Fibonacci numbers in a *unique* way. [*Hint: consider the smallest number that admits two decompositions; what can you say about the largest Fibonacci numbers appearing in these two decompositions?*]

8.5 Let $a_1 := 5$, $a_2 := 19$, and $a_{n+1} := 5a_n - 6a_{n-1}$ for all integers $n \geq 2$. Use the method of generating functions to conjecture an explicit formula for a_n.