Instructor: Leo Goldmakher

NAME (LAST, NICKNAME): ____________________

SECTION # (10AM = 1, 11AM = 2): ____________

Williams College
Department of Mathematics and Statistics

MATH 200 : DISCRETE MATH

Problem Set 9 – due Thursday, April 25th

INSTRUCTIONS:
This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Late assignments may be submitted at the beginning of Friday's class to me in person (i.e. don't leave them in my mailbox or ask someone else to submit on your behalf); however, 5% will be deducted for late submission. *Assignments submitted later than start of class on Friday will not be graded.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
9.1	
9.2	
9.3	
9.4	
9.5	
Total	

Please read the following statement and sign **before writing the final version of this problem set:**

I understand that I am not allowed to use the internet to assist with this assignment, apart from accessing the course website or looking up definitions. I also understand that I must write down the final version of my assignment in isolation from any other person, and to not copy from any set of written notes created when another person was present. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________________

Problem Set 9

9.1 Use the Doomsday algorithm to calculate the day of the week of the following dates. You don't have to show much work, but do write down all five relevant numbers that come up in the algorithm (the doomsday number, the century day, the year over 12, the remainder, the remainder over 4).

(a) June 6, 1944 (D-day)

(b) November 19, 1863 (Gettysburg address)

(c) January 21, 1793 (Execution of Louis XVI)

(d) February 22, 1980 (Miracle on Ice)

9.2 Prove that any positive integer n can be expressed as a sum of powers of 3 where each power of 3 appears at most twice. For example, $59 = 3^3 + 3^3 + 3^1 + 3^0 + 3^0$. (This is called the ternary expansion of n.)

9.3 Write down the multiplication table (excluding the 0 row and columns) for arithmetic (mod 12).

9.4 The goal of this problem is to establish the following remarkable result:

Bézout's theorem. If $a, b \in \mathbb{Z}_{>0}$, then $\exists x, y \in \mathbb{Z}$ such that $\gcd(a, b) = ax + by$.

Here $\gcd(a, b)$ denotes the greatest common divisor of a and b (i.e. the largest positive integer that divides both a and b). Throughout this problem, we'll use the notation

$$a\mathbb{Z} + b\mathbb{Z} := \{ax + by : x, y \in \mathbb{Z}\}.$$

(a) Write down five numbers that live in $2\mathbb{Z} + 3\mathbb{Z}$. What's a simpler name for the set $2\mathbb{Z} + 3\mathbb{Z}$? Prove it. [*Hint: as a warm-up, show that if $n \in 2\mathbb{Z} + 3\mathbb{Z}$ then $17n \in 2\mathbb{Z} + 3\mathbb{Z}$. (This is just for your scratch-work.)*]

(b) Given $a, b \in \mathbb{Z}_{>0}$, let d denote the smallest positive element of $a\mathbb{Z} + b\mathbb{Z}$. Prove that $d \mid a$ and $d \mid b$. [*Hint: by the quotient-remainder theorem, we can write $a = qd + r$. What can you deduce about r?*]

(c) Given $a, b \in \mathbb{Z}_{>0}$, let d denote the smallest positive element of $a\mathbb{Z} + b\mathbb{Z}$. Suppose $c \in \mathbb{Z}_{>0}$ is a common divisor of a and b, i.e. $c \mid a$ and $c \mid b$. Prove that $c \mid d$.

(d) Prove Bézout's theorem (stated in the introduction to this problem). Your proof should be very short.

9.5 In this problem, you will explore some divisibility rules.

(a) Prove that $n \in \mathbb{Z}_{>0}$ is a multiple of 3 if and only if the sum of the digits of n is a multiple of 3. [*Hint: any number can be expressed in the form $10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$, where $a_i \in \{0, 1, \ldots, 9\}$ for each i.*]

In the next two parts you will explore a divisibility rule for 7. Given a positive integer n with k digits, form a new number $f_7(n)$ as follows: split off the last (rightmost) digit of n, double it, and subtract it from the number formed by the first $k - 1$ digits of n. The resulting number is what I call $f_7(n)$. I claim that $7 \mid n$ iff $7 \mid f_7(n)$. For example, is 8526 a multiple of 7? Split off the last digit (6), double it (12), and subtract it from the number formed by the other digits ($852 - 12 = 840$). So, $f_7(8526) = 840$, and the divisibility rule asserts that 8526 is a multiple of 7 iff 840 is. Now we can repeat the same procedure for 840: split off and double the last digit, and subtract from the other digits to find that $f_7(840) = 84 - 0 = 84$. Finally, $f_7(84) = 8 - 2 \times 4 = 0$, which is definitely a multiple of 7. It follows that 84 is as well, hence so is 840, hence so is 8526. (Note that if you recognized 840 as a multiple of 7, you could have stopped the process there.)

(b) Use the above divisibility rule to determine (by hand!) whether or not 295781 is a multiple of 7.

(c) Prove that $7 \mid n$ iff $7 \mid f_7(n)$. [*Hint: write down an algebraic formula connecting n and $f_7(n)$, and consider it (mod 7).*]

(d) Formulate and prove a divisibility rule for 13. [*Please don't look this up! Instead, play around with the method for divisibility by 7 and see if you can adapt it to 13.*]