Instructor: Leo Goldmakher

Name (Last, Nickname): ______________________

Section # (10am = 1, 11am = 2): ______________

Williams College
Department of Mathematics and Statistics

MATH 200 : DISCRETE MATH

Problem Set 11 – due Thursday, May 9th

INSTRUCTIONS:

This assignment must be turned in to my mailbox (on the right as you enter Bascom) by **4pm** sharp. Late assignments may be submitted at the beginning of Friday's class to me in person (i.e. don't leave them in my mailbox or ask someone else to submit on your behalf); however, 5% will be deducted for late submission. *Assignments submitted later than start of class on Friday will not be graded.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
11.1	
11.2	
11.3	
11.4	
11.5	
11.6	
Total	

Please read the following statement and sign **before writing the final version of this problem set:**

I understand that I am not allowed to use the internet to assist with this assignment, apart from accessing the course website or looking up definitions. I also understand that I must write down the final version of my assignment in isolation from any other person, and to not copy from any set of written notes created when another person was present. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 11

11.1 From now on we define

$$\binom{n}{k} := 0 \qquad \text{whenever } 0 \leq n < k. \tag{$\dagger$}$$

This definition allows us to extend Pascal's triangle arbitrarily far to the right to form 'Pascal's rectangle':

$$\begin{array}{lccccccccc}
\text{(0th row)} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\text{(1st row)} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\text{(2nd row)} & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\
 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & \cdots \\
 & 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & \cdots \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & \cdots \\
 & & & & & \vdots & & & &
\end{array}$$

(a) Why is the definition ($\dagger$) reasonable? In other words, why is it logically consistent with our definition of $\binom{n}{k}$ for $0 \leq k \leq n$?

(b) Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ whenever $k, n \in \mathbb{Z}_{\geq 0}$. [*Hint: when $b \leq a$ we have a formula for $\binom{a}{b}$, and when $b > a$ we can use ($\dagger$).*]

(c) Fix an integer $n \geq 0$. The n^{th} row of Pascal's rectangle is the infinite sequence of numbers

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \cdots$$

What is the generating function of this sequence? [*Your answer should be a finite expression. For example, $\frac{1}{1-x-x^2}$ is the generating function of the Fibonacci numbers.*]

(d) Use our recursive formula (proved in **11.1**b) to write the -1^{st}, -2^{nd}, and -3^{rd} rows of Pascal's rectangle. Are these consistent with what you would expect for the generating function of these rows (from your work in **11.1**c)?

(e) (**For fun, not to be turned in.**) What should the ($1/2$)-th row of Pascal's rectangle be?

11.2 We explore some nice properties of the binomial coefficients.

(a) Discover and prove a formula for $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots$. [*Your proof might be very short!*]

(b) Discover and prove a formula for $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \binom{n}{8} + \cdots$ [*Hint: use the previous part.*]

(c) Discover and prove a formula for $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \cdots$ [*Hint: play with $(1+x)^n$*]

(d) Prove that $\frac{1}{n+1}\binom{2n}{n}$ is an integer for every positive integer n.

(e) Discover a formula for

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

You don't have to prove it.

11.3 In the most common version of poker, each player is dealt five cards out of a standard 52-card deck; the set of five cards is called a *poker hand*.[1] There are $\binom{52}{5} = 2598960$ different five-card hands. (Note that two poker hands are the same if they consist of the same five cards, the order of the cards in a poker hand is not important.)

(a) A hand is called *four-of-a-kind* if it contains four cards of the same rank. For example, the hand

$$3\spadesuit, 3\heartsuit, 3\diamondsuit, 3\clubsuit, K\spadesuit$$

is a four-of-a-kind. How many different four-of-a-kind hands are there? [*Hint: how many possible ranks of the four matching cards are there? Once you've selected the four matching cards, how many possibilities remain for the fifth card?*]

(b) A hand is called a *full house* if it contains three cards of one rank and two cards of another rank. For example, the hand

$$3\spadesuit, 3\heartsuit, K\spadesuit, K\diamondsuit, K\clubsuit$$

is a full house. How many different full house hands are there?

11.4 Recall the generating function of the Fibonacci numbers was

$$F(x) = \frac{1}{1 - x - x^2}$$

(a) Write down an infinite geometric series whose sum is $F(x)$. [*Recall that a geometric series is a sum of the form* $1 + r + r^2 + r^3 + \cdots$.]

(b) Use part (a) to deduce a formula for f_n (the nth Fibonacci number) in terms of binomial coefficients.

11.5 Prove that $\binom{p}{k}$ is a multiple of p for any prime p and any integer $k \in \{1, 2, \ldots, p-1\}$.

11.6 Exploring the four-coloring theorem!

(a) Give a four-coloring of the map of the continental US (see next page). [*You can use real colors, or you can mark each state with a number 1–4.*]

(b) Does there exist a three-coloring of the continental US? If so, draw it on the map. If not, explain why not.

[1] A deck of cards consists of 52 cards, with each card assigned a unique *rank* and *suit*. There are thirteen ranks and four suits:

Ranks: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King
Suits: $\heartsuit$ (hearts), $\diamondsuit$ (diamonds), $\clubsuit$ (clubs), $\spadesuit$ (spades)

$2\heartsuit$ ('the two of hearts') and $Q\spadesuit$ ('the queen of spades') are both examples of cards.