LINEAR ALGEBRA: LECTURE 4

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Given $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) \qquad \forall x, y \in \mathbb{R}. \tag{1}$$

We previously proved the following:

Theorem 1. *$f(\alpha) = \alpha f(1)$ for every rational α.*

What about irrational α? For example, what can we say about $f(\sqrt{2})$? After some playing around, we realized that every real number can be written as a sum of rational numbers. For example,

$$\sqrt{2} = 1 + 0.4 + 0.01 + 0.004 + \cdots$$

One might therefore try to use additivity to figure out $f(\sqrt{2})$:

$$\begin{aligned} f(\sqrt{2}) &= f(1 + 0.4 + 0.01 + 0.004 + \cdots) \\ &= f(1) + f(0.4) + f(0.01) + f(0.004) + \cdots \\ &= f(1) + 0.4f(1) + 0.01f(1) + 0.004f(1) + \cdots \\ &= (1 + 0.4 + 0.01 + 0.004 + \cdots)f(1) \\ &= \sqrt{2}\, f(1). \end{aligned} \tag{2}$$

This looks legitimate. However, there's a subtle but very important issue in this chain of equalities. To explain this, we need a bit of notation. Write

$$\sqrt{2} = \sum_{n=0}^{\infty} a_n \tag{3}$$

where $a_n \in \mathbb{Q}$ for every n; above, we choose $a_0 = 1$, $a_1 = 0.4$, $a_2 = 0.01$, $a_3 = 0.004$, etc. What does the statement (3) actually mean? Are we really adding up infinitely many terms? I, for one, have no idea how to accomplish this. (I might not be able to sum up a million numbers, but I know exactly what it would entail. However, I do not understand how one would go about adding infinitely many numbers together.) After some discussion, we decided that (3) is shorthand for the more complicated idea

$$\sqrt{2} = \lim_{N\to\infty} \sum_{n=0}^{N} a_n,$$

which itself is shorthand for the even more complicated idea that the *finite* sum on the right hand side gets arbitrarily close to $\sqrt{2}$ as N gets larger and larger. (In particular, the $=$ sign in (3) is a red herring – that sum isn't a literal sum of infinitely many terms, nor do any of the finite sums it's representing equal $\sqrt{2}$.) In short, the second equality in our chain of equalities (2) can be expanded and rewritten in the form

$$\begin{aligned} f\Big(\lim_{N\to\infty} \sum_{n=0}^{N} a_n\Big) &= \lim_{N\to\infty} f\Big(\sum_{n=0}^{N} a_n\Big) \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} f(a_n). \end{aligned}$$

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The second equality here is straightforward – it follows easily from (1) – but the first step of pulling the limit out from inside the function is dubious. Some of you recognized the process of pulling a limit out from inside a function as a statement about continuity. More precisely, recall that saying f is continuous at $\sqrt{2}$ means

$$\lim_{x \to \sqrt{2}} f(x) = f\big(\sqrt{2}\big).$$

Thus, if f is continuous at $\sqrt{2}$, then pulling the limit out of f is OK, so the computation (2) works! If, however, f is *not* continuous, then (2) will be wrong – to be discontinuous precisely means that when you try to pull the limit out of the function, wonky stuff happens. Summing up, we've proved the following:

Theorem 2. *Any* continuous *function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* (1) *must also satisfy*

$$f(\alpha) = \alpha f(1) \qquad \forall \alpha \in \mathbb{R}.$$

This leads us to a natural query:

Question. *Do there exist discontinuous additive functions* $f : \mathbb{R} \to \mathbb{R}$*?*

The answer to this question is somewhat unsatisfying: if you believe the Axiom of Choice, then the answer is YES, while if you do not believe in the Axiom of Choice, there is no definite answer. We spent some time discussing the Axiom of Choice and the standard (Zermelo-Fraenkel) axioms of set theory. I urge you to read about the fascinating work of Kurt Gödel and Paul Cohen on the independence of the Axiom of Choice from the other axioms on Wikipedia. (Under the article on *Axiom of Choice*, check out the section *Independence*.)

Next class we will begin discussing what is usually called linear algebra. As you shall see, it bears more than a passing resemblance to what we've been exploring thus far.