LINEAR ALGEBRA: LECTURE 5

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Thus far, I haven't told you what linear algebra is. Here's the shortest description:

Linear algebra is the study of linear maps.

This is, of course, totally unhelpful without knowing what a linear map is. Rather than telling you what a linear map is, we'll explore a few easy examples of linear maps. Here's the simplest type of linear map:

Definition. A *linear map from* $\mathbb{R}$ *to* $\mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying both of the following properties:

(1) f is *additive*, i.e., $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
(2) f *scales*, i.e., $f(\alpha x) = \alpha f(x)$ for all $\alpha, x \in \mathbb{R}$.

From our prior work, additivity implies something similar to scaling, but only for $\alpha \in \mathbb{Q}$; for linearity we are more demanding, requiring that f scale for *every* $\alpha \in \mathbb{R}$. Note that we do *not* require a linear map to be continuous.

What are some examples of linear maps from $\mathbb{R}$ to $\mathbb{R}$?

(1) $f(x) = 3x$ is linear.
   - f is additive: $f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y)$.
   - f scales: $f(\alpha x) = 3\alpha x = \alpha \cdot 3x = \alpha f(x)$
(2) More generally, $f(x) = kx$ is linear for any $k \in \mathbb{R}$. (Verify this!)
(3) $f(x) = 3x + 2$ is *not* linear: it is neither additive nor does it scale. (Check!)

After some more playing around, we conjectured that every linear map from $\mathbb{R}$ to $\mathbb{R}$ is of the form kx. In other words, not only is every function of the form kx linear, but also every linear map is of the form kx. We can combine these two statements into one using the useful phrase *if and only if*:

Proposition 1. *Given a function* $f : \mathbb{R} \to \mathbb{R}$*. Then* f *is linear if and only if* $f(x) = kx$ *for some* $k \in \mathbb{R}$*.*

Proof. The statement of the proposition consists of two independent assertions:

($\Rightarrow$) *If* f *is linear, then* $f(x) = kx$*.*
($\Leftarrow$) *If* $f(x) = kx$*, then* f *is linear.*

We prove these statements one at a time.
($\Rightarrow$)

Suppose f is linear. In particular, f scales, which implies (taking $x = 1$) that $f(\alpha) = \alpha f(1)$ for any $\alpha \in \mathbb{R}$. This shows that $f(x) = kx$, with the constant k being $f(1)$.

($\Leftarrow$)

Suppose $f(x) = kx$. It's easy to verify that f is additive and scales, hence is linear. $\square$

Note that the function $f(x) = 3x + 2$, which we might normally call a linear function, is *not* a linear map. This seems totally unreasonable at first, yet there is method in it. To explain this, consider the following picture of a line:

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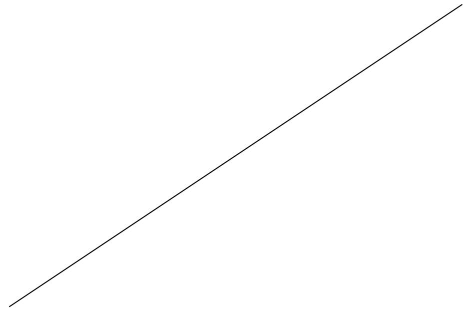

A picture of a line

What is the equation of this line? Without coordinate axes, it's impossible to say. And in both math and life, we often *don't* have axes given to us. Fortunately, in such a situation, it doesn't matter what the axes are. The purpose of an equation is to describe the line, and we can do it in any number of ways. One way is to add your own set of axes to the picture, figure out the equation of the line, and make that your description. Then your friend can graph the line from your equation (using their own set of axes), erase the axes and voilà! They have the same picture as you. Note that by drawing the axes in such a way that the line passes through the origin, we get the simplest description of the line: it will be of the form $f(x) = kx$. This type of coordinate-free approach will be a recurring theme throughout our course.

Thus, we see that linear algebra in the world of functions $\mathbb{R} \to \mathbb{R}$ is trivial: the linear maps are precisely the functions $f(x) = kx$. Let's bump up the difficulty a tiny bit, to involve the coordinate plane $\mathbb{R}^2$. Here is a formal definition of $\mathbb{R}^2$:

$$\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}.$$

With this in hand, we can now state the next simplest example of a linear map after the example we considered above.

Definition. A *linear map from* $\mathbb{R}^2$ *to* $\mathbb{R}$ is a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying both of the following properties:

(1) f is *additive*, i.e., $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^2$.
(2) f *scales*, i.e., $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$.

This seems simple enough, but what do these properties actually mean? How does one add two points in the plane? Exactly the way you would guess: $(a, b) + (c, d) := (a + c, b + d)$. What about scaling a point by α? Again, the natural guess turns out to be the correct one: $\alpha(x, y) := (\alpha x, \alpha y)$.

What can we say about linear maps $\mathbb{R}^2 \to \mathbb{R}$? It's easy to generate examples:

(1) $f(x, y) = 3x - 2y$ is linear: it's easy to check that it's additive and scales.
(2) More generally, $f(x, y) = ax + by$ is linear for any choices of a and b. (Verify!)
(3) $f(x, y) = x + y + 1$ is not linear: it doesn't scale, nor is it additive.

At this point, we conjectured that the only linear maps from $\mathbb{R}^2$ to $\mathbb{R}$ were those of the form $ax + by$ for some constants a and b. This is indeed the case:

Theorem 2. *A function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is linear if and only if there exist* $a, b \in \mathbb{R}$ *with* $f(x, y) = ax + by$.

Proof. As above, the statement (being if and only if) is secretly two independent statements. We prove each separately.

($\Rightarrow$)

Suppose f is linear. Set $a := f(1, 0)$, and $b := f(0, 1)$. Then we have

$$\begin{aligned} f(x, y) &= f(x, 0) + f(0, y) && \text{(by additivity)} \\ &= x f(1, 0) + y f(0, 1) && \text{(by scaling)} \\ &= ax + by. \end{aligned}$$

($\Leftarrow$)

Suppose $f(x, y) = ax + by$. Then it is easy to verify that f is additive and scales, hence is linear. $\square$