LINEAR ALGEBRA: LECTURE 9

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Last time we found a formula for the rotation map:

$$R_\theta(x,y) = \Big(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\Big). \tag{1}$$

An immediate consequence is the following:

Corollary 1. *R_θ is a linear map.*

Recall that linear maps from $\mathbb{R} \to \mathbb{R}$ and from $\mathbb{R}^2 \to \mathbb{R}$ had similar forms: in both cases, a function f was linear iff $f(x) = k \cdot x$. Can we extend this to linear maps from $\mathbb{R}^2 \to \mathbb{R}^2$? In other words, is there some notion of multiplication such that any linear $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be written in the form $f(x) = k \cdot x$ for every $x \in \mathbb{R}^2$? We start with a more concrete question. Can R_θ be written in the form $R_\theta(x) = k \cdot x$? We quickly realized that in the formula (1) each coordinate was a dot product:

$$x\cos\theta - y\sin\theta = (\cos\theta, -\sin\theta)\cdot(x,y)$$
$$x\sin\theta + y\cos\theta = (\sin\theta, \cos\theta)\cdot(x,y)$$

We therefore assemble these 'coefficients' into a single 2×2 matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, and say

$$R_\theta(x,y) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot(x,y)$$

There's a certain asymmetry in this 'multiplication' – the dot product of the top row of the matrix with (x, y) becomes the left coordinate, while the dot product of the bottom row of the matrix with (x, y) becomes the right coordinate. We rectify this by writing points of $\mathbb{R}^2$ vertically rather than horizontally.

Convention. *Henceforth the symbol $\begin{pmatrix}x\\ y\end{pmatrix}$ will be used as an alternative notation to the standard (x, y).*

Thus we write

$$R_\theta\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}$$

where

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} := \begin{pmatrix}ax+by\\ cx+dy\end{pmatrix}$$

Note that the dot product of the top row of the matrix with the point $\begin{pmatrix}x\\ y\end{pmatrix}$ yields the top row (coordinate) of the resulting point, while the dot product of the bottom row of the matrix with the point $\begin{pmatrix}x\\ y\end{pmatrix}$ yields the bottom row (coordinate) of the resulting point.

We will abuse notation similarly to how we did before, and write

$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

For example, we have

$$R_{\pi/2} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Date: February 26, 2016.

More concretely, this means that

$$R_{\pi/2}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}$$

More generally, we can think of any matrix as a function. In fact, it is a linear function:

Theorem 2. *A function* $f:\mathbb{R}^2\to\mathbb{R}^2$ *is linear iff* $f=\begin{pmatrix}a&b\\c&d\end{pmatrix}$.

Proof. We treat the two implications separately.

($\Rightarrow$) Given a linear map $f:\mathbb{R}^2\to\mathbb{R}^2$, write

$$f\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}\qquad\text{and}\qquad f\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}b\\d\end{pmatrix}$$

By linearity, we have

$$f\begin{pmatrix}x\\y\end{pmatrix}=f\begin{pmatrix}x\\0\end{pmatrix}+f\begin{pmatrix}0\\y\end{pmatrix}=xf\begin{pmatrix}1\\0\end{pmatrix}+yf\begin{pmatrix}0\\1\end{pmatrix}=x\begin{pmatrix}a\\c\end{pmatrix}+y\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}ax+cy\\bx+dy\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$

whence $f=\begin{pmatrix}a&b\\c&d\end{pmatrix}$.

($\Leftarrow$) Given $f=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ it is a straightforward exercise to verify that f must scale and be additive. □

Note that in the proof, we found a way to explicitly find the matrix of f. First, determine the two points $f\begin{pmatrix}1\\0\end{pmatrix}$ and $f\begin{pmatrix}0\\1\end{pmatrix}$. The matrix whose columns are these two points is the matrix of f! For example, let $\rho:\mathbb{R}^2\to\mathbb{R}^2$ be the function which reflects points across the horizontal axis. Then $\rho\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\rho\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$, whence

$$\rho=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$