LINEAR ALGEBRA: LECTURE 10

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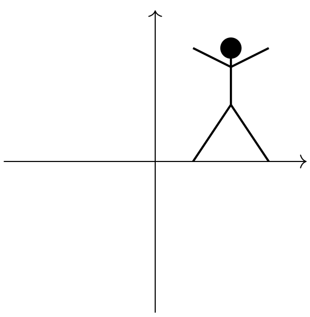
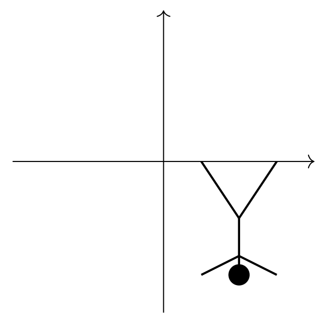
Today we discussed the geometric nature of linear maps. We applied a bunch of linear maps to the stick figure below to see the geometry in action:

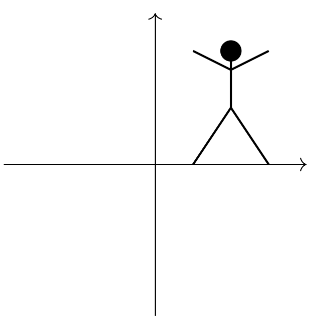
Reflection across horizonal axis



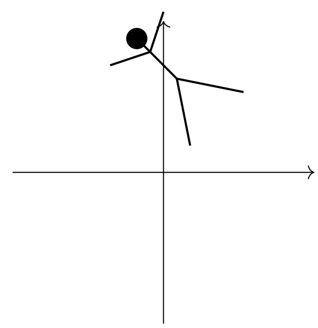
$$\xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$



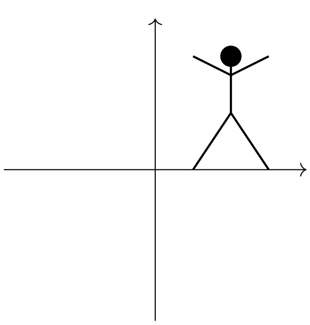
Rotation by $\pi/4$



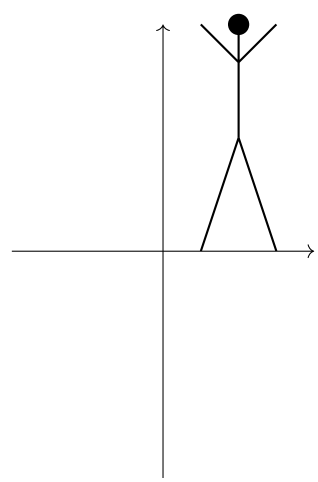
$$\xrightarrow{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}}$$



Vertical stretch

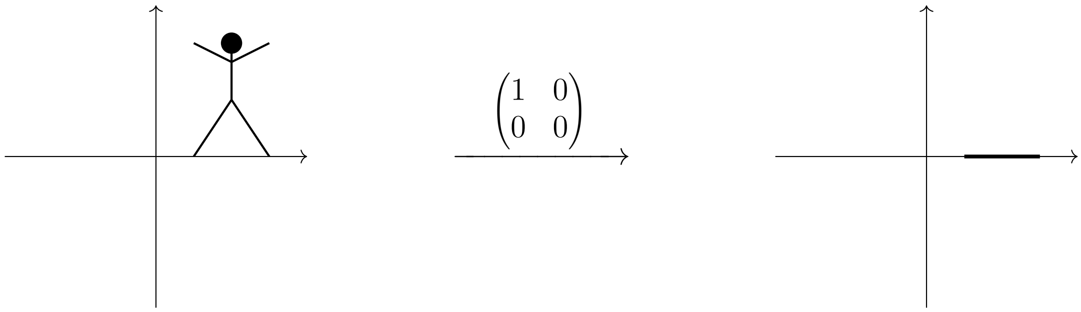


$$\xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}$$



Date: February 29, 2016.

SQUASHING



We can also create more complicated transformations by applying one linear map, followed by another, followed by another, etc. The following definition formalizes this idea:

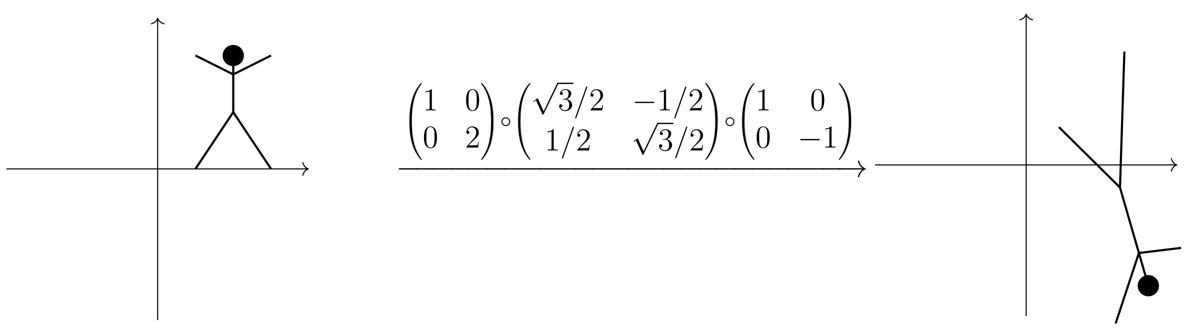
Definition. Given two functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$. The *composition* of f and g, denoted $f \circ g$, is the function mapping $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(f \circ g)(x) := f\Big(g(x)\Big).$$

One word of caution: composition reverses the natural order in which we list the maps. For example, if we start with our stick figure and apply $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$, and finally $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then altogether we've applied the map

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \circ \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Make sure you can explain why the order is the reverse of how we stated it in words! Below is a picture of the action of this particular map on our poor stick figure:



This brings us to one of the fundamental questions of linear algebra.

Question. *Given a linear map f. Is there a simple description of what f does geometrically?*

For example, the map $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$ rotates the plane by an angle of $\pi/6$ counterclockwise about the origin. What does the matrix $\begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}$ do? Of course we can find a formula for where it sends any particular point, but is there an easy-to-comprehend qualitative description?

Before tackling this question, we warm up with an easier one: what can we say about the set of all outputs of a given linear map? This is called the *image* of the map. Here's a formal definition:

Definition. Given a function $f : A \to B$. The *image* of f is defined to be

$$\operatorname{im}(f) := f(A) := \{f(a) : a \in A\}.$$

Example 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then $\text{im}\,(f) = \mathbb{R}_{\geq 0}$, the set of all non-negative real numbers.

Example 2. The image of the rotation map $R_{\pi/2}$ is

$$\text{im}\,(R_{\pi/2}) = \mathbb{R}^2.$$

Example 3. The image of the squashing map from above is the horizontal axis:

$$\text{im}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \{(x, 0) : x \in \mathbb{R}\}.$$

Next lecture, we will explore more carefully what we can say about the images of general linear maps.