LINEAR ALGEBRA: LECTURE 13

LEO GOLDMAKHER

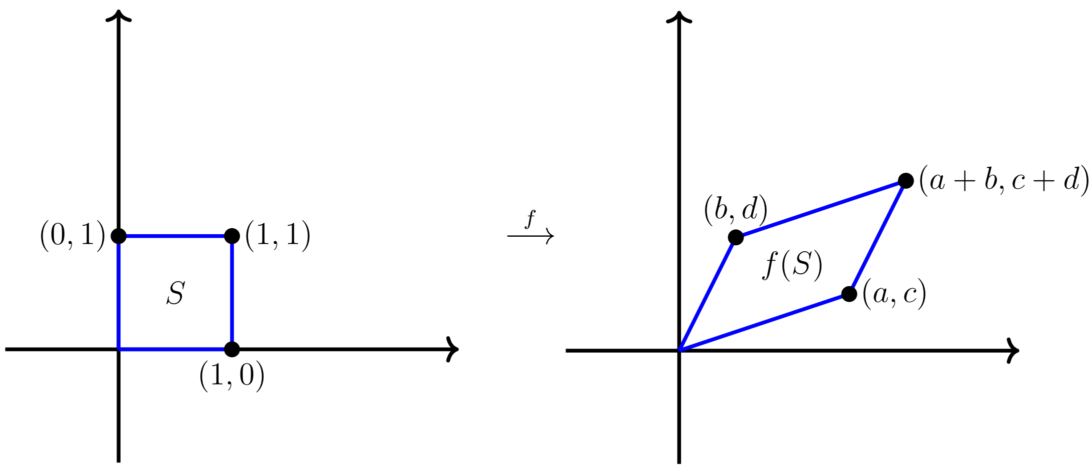
We've seen how linear maps can affect stick figures: they can rotate them, stretch them, squash them, etc. Today we examine a related question:

Question. *How does a linear function affect the area of a shape?*

Let's start with the most basic version of this question. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function. What does it do to a unit of area? Let S denote the unit square whose lower left corner is at the origin. (See figure below.) Since f is linear, we can write it as a matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

What's the image of S under f?



(One way to see this is to recognize the first column of the matrix as $f(1,0)$ and the second column as $f(0,1)$, and then to use additivity to figure out $f(1,1)$. Alternatively, we can use matrix multiplication.)

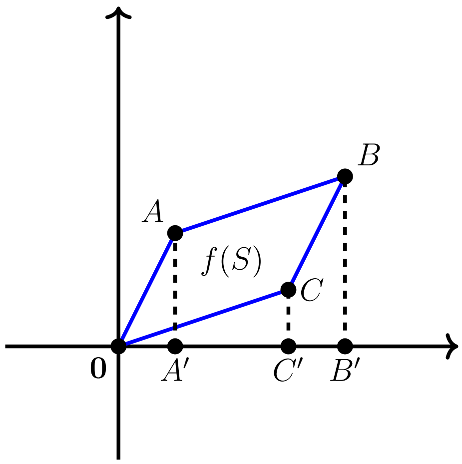
After a bit of work, we were able to compute the area of $f(S)$:

$$\text{area } f(S) = ad - bc.$$

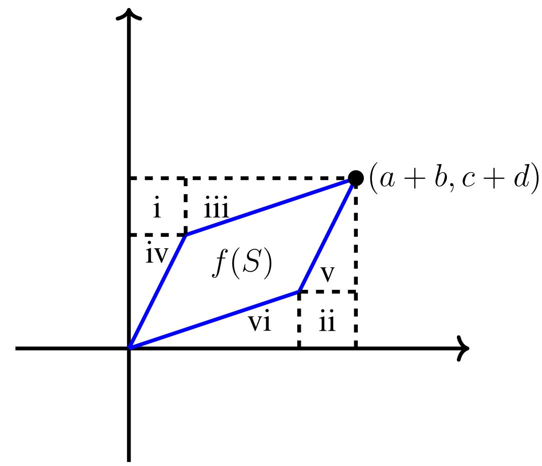
There are a few different ways to do this. One approach is to use the dot product to calculate the angle between (a, c) and (b, d), and then to use this to calculate the volume. A related idea is to evaluate the area as base multiplied by height; for example we could take the segment connecting $\mathbf{0}$ to (a, c) as the base, and then the perpendicular from (b, d) to this segment as the height. This is related because finding the height involves figuring out the angle between (a, c) and (b, d). A third idea was to view (a, c) and (b, d) as the 3-dimensional vectors $(a, c, 0)$ and $(b, d, 0)$ sitting in the plane, and then use their cross product to calculate the area.

These are fine ideas, but are all a bit complicated – they either involve remembering some tricky formulas, or else require a fair amount of computation. To get around this, in class we came up with a couple of simpler approaches. Both involved breaking up $f(S)$ into simple shapes, like right triangles, right trapezoids, and rectangles. Here are the two we came up with:

Date: March 7, 2016.



area $f(S) =$
area $(\mathbf{0}AA') +$ area $(AA'B'B) -$
area $(\mathbf{0}CC') -$ area $(BCC'B')$



area $f(S) =$
$(a+b)(c+d) -$ area $(i) -$ area $(ii) -$
area $(iii) -$ area $(iv) -$ area $(v) -$ area (vi)

Two approaches to calculating the area of $f(S)$

Following either approach[1] led us to deduce

$$\text{area } f(S) = ad - bc.$$

Since the quantity $ad - bc$ has come up several times already, it deserves a name:

Definition. The *determinant* of a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\det f := ad - bc$ where $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Above we proved

Proposition 1. *Given $f : \mathbb{R}^2 \to \mathbb{R}^2$ a linear map. Then*

$$\text{area } f(S) = \det f$$

where S is the unit square illustrated above.

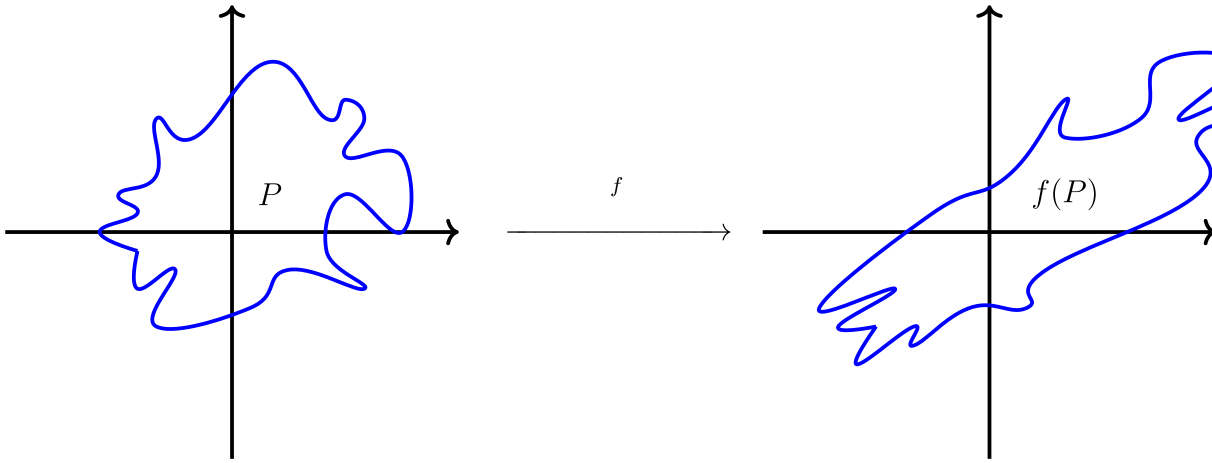
One aspect of this proposition is a little fishy: the determinant of f might be negative, while area traditionally is not! There are two ways to fix this. One approach – which you will think about on your homework this week – is a way to view area which allows it to be negative. A different (cruder) fix is to change the conclusion of the proposition to

$$\text{area } f(S) = |\det f|.$$

For reasons which will become apparent, the first approach is better, so I will continue to use Proposition 1 in its current incarnation.

Why is $ad - bc$ called the *determinant*? Consider an arbitrary shape P in the plane and its image under some linear map f:

[1] The first approach, in particular, is reminiscent of a Williams-related result: President Garfield's proof of the Pythagorean Theorem, which computes the area of a certain trapezoid in two different ways.



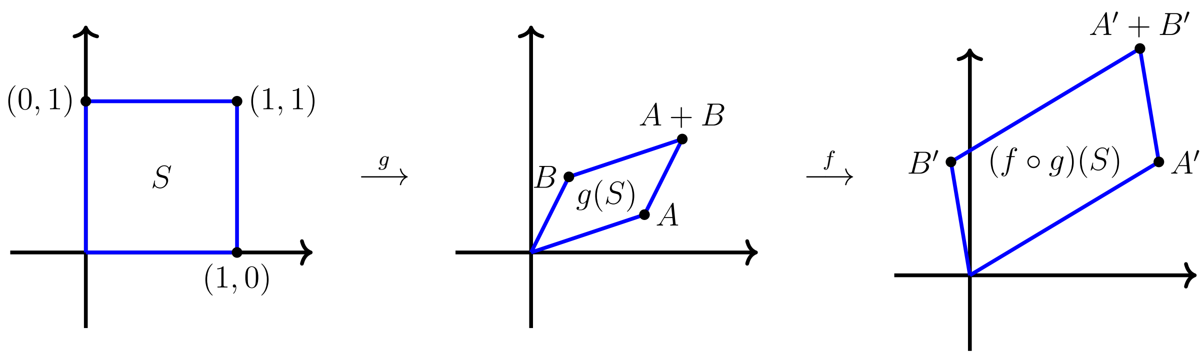
Is there a nice relationship between P and $f(P)$?

Of course the original shape and its image look quite different. However, it turns out there's a beautiful relationship between the areas of the two:

$$\text{area } f(P) = (\det f) \cdot \text{area } P. \tag{1}$$

In other words, the determinant of f determines (!) how area scales under f. We've already seen one special case of this – Proposition 1 – and we will prove a few other special cases of this. However, we will not prove this fact in general, because it's quite complicated.[2]

Although we won't prove this property of the determinant, we will use it as inspiration for things we can prove about the determinant. For example, given two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, what can one say about $\det(f \circ g)$? Consider the following illustration:



Start with the unit square, then view its images under two successive linear transformations

As we discussed earlier, the image of the unit square S under a linear map must be a parallelogram: if A is the image of $(1, 0)$ and B is the image of $(0, 1)$, then the location of the image of $(1, 1)$ is determined by additivity. For the same reason, the image of any parallelogram must also be a parallelogram. By Proposition 1 we know that

$$\text{area } g(S) = \det g \qquad \text{and} \qquad \text{area } \Big((f \circ g)(S)\Big) = \det(f \circ g)$$

On the other hand, by the (unproved) relation (1) we expect

$$\text{area } \Big((f \circ g)(S)\Big) = (\det f) \cdot \text{area } g(S).$$

Putting all these together leads us to the following guess:

Conjecture 2. *Given two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$. Then*

$$\det(f \circ g) = (\det f)(\det g).$$

[2] In fact, even defining what area means exactly is complicated. There exist shapes in the plane which have no well-defined area! You can learn more about this in a course on measure theory.