LINEAR ALGEBRA: LECTURE 14

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We began by proving our conjecture from last time.

Proposition 1. *Given two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$. Then*

$$\det(f \circ g) = (\det f)(\det g).$$

Proof. Since f and g are linear, we can write them as matrices, say,

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} \ell & m \\ n & p \end{pmatrix}$$

It follows from our earlier work that $f \circ g$ is also a linear map, and has the matrix

$$f \circ g = \begin{pmatrix} a\ell + bn & am + bp \\ c\ell + dn & cm + dp \end{pmatrix}$$

Thus we have

$$\det f = ad - bc \qquad \det g = \ell p - mn \qquad \det(f \circ g) = (a\ell + bn)(cm + dp) - (am + bp)(c\ell + dn)$$

A straightforward computation verifies the identity $\det(f \circ g) = (\det f)(\det g)$. $\square$

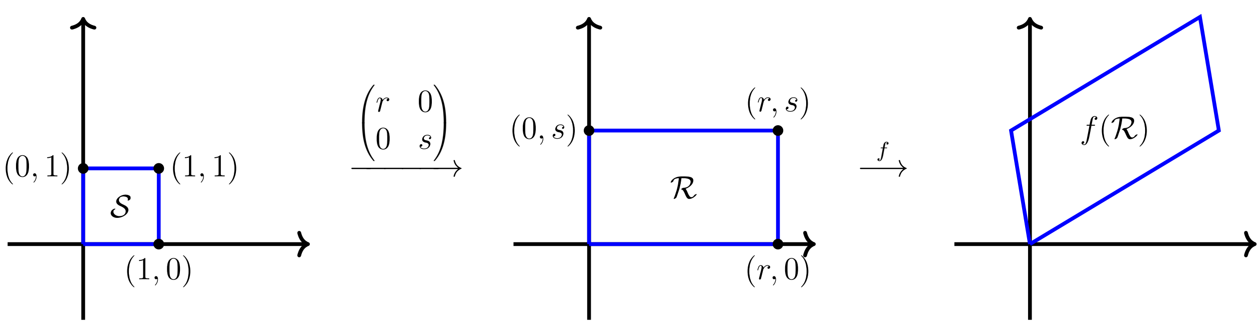
Recall from last lecture that we asserted (without proof) that the quantity $\det f$ determines how f scales area. Thus far we've rigorously proved only one special case of this: that the image under f of S (the unit square with lower left corner at the origin) has area $\det f$. We now use the proposition above to generalize this a bit:

Proposition 2. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map. Then*

$$\text{area } f(\mathcal{R}) = (\det f) \cdot \text{area } \mathcal{R}$$

where $\mathcal{R}$ denotes the $r \times s$ rectangle with lower left corner at the origin.

The proof is most easily understood via the illustration below:



Proof of Proposition. We first observe that $\mathcal{R}$ is the image of the unit square $\mathcal{S}$ under a pretty simple linear map:

$$\mathcal{R} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \mathcal{S}.$$

It follows that

$$f(\mathcal{R}) = f \circ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \mathcal{S}.$$

Date: March 9, 2016.

Using Proposition 13.1 and Proposition 1, we see that

$$\text{area } f(\mathcal{R}) = \det\left(f \circ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\right) = (\det f) \cdot \det \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$
$$= (\det f) \cdot rs = (\det f) \cdot \text{area } \mathcal{R} \qquad \square$$

Next, we turned to a new topic:

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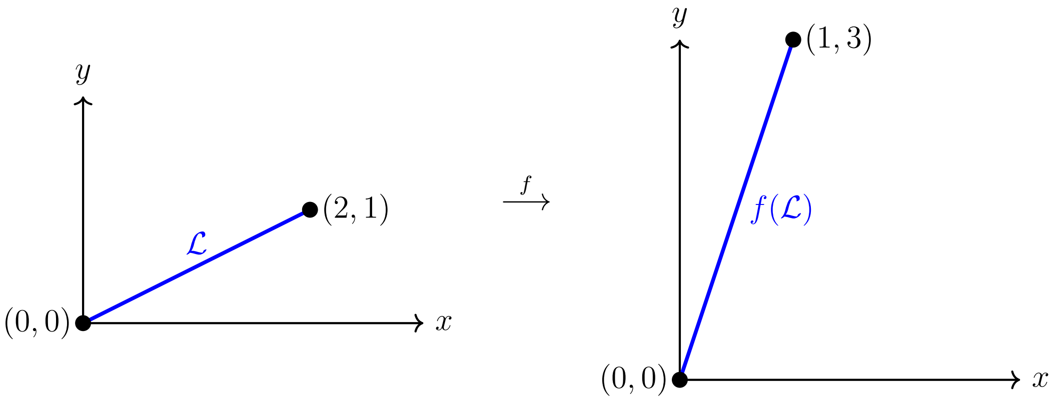
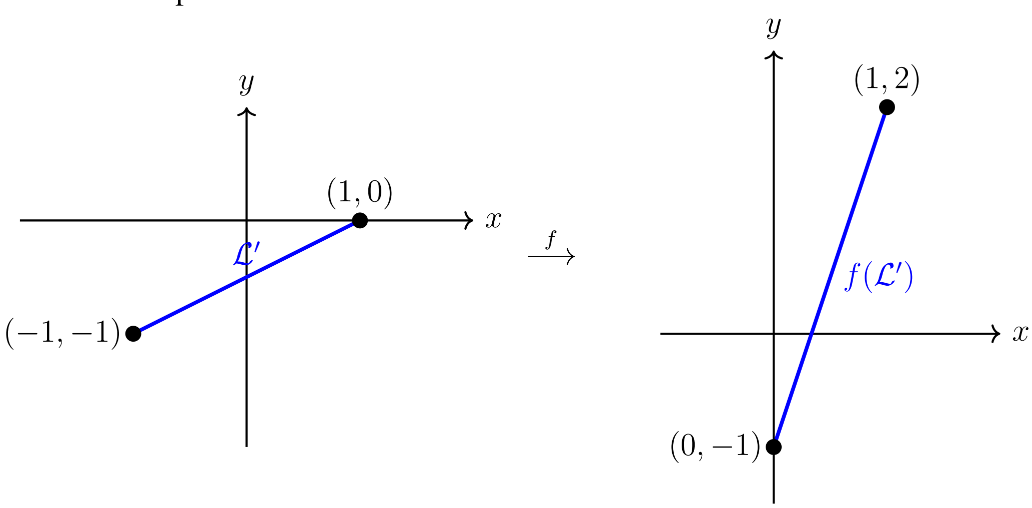
Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) := (x - y, 2x - y).$$

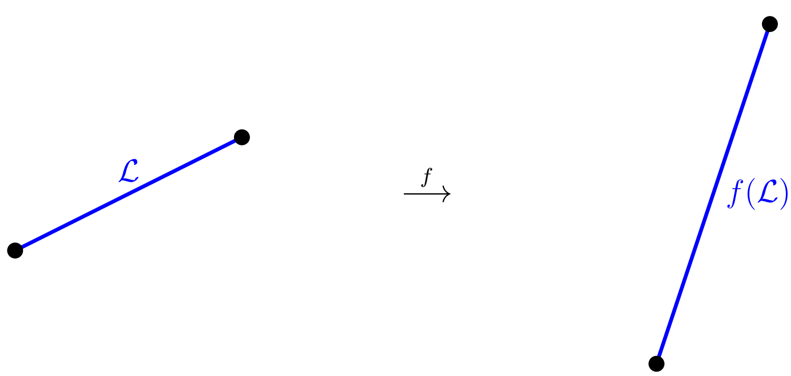
From your homework, you know that f maps line segments to line segments. For example:



Or another example:



The curious thing about the above is that if we remove the coordinate axes, both of them look like this:



Having removed the axes (and therefore any hope of labeling the endpoints), it's a bit trickier to describe $\mathcal{L}$. Certainly, we can no longer write $\mathcal{L}$ as a set of points, since we have no way to label them! How else can we describe it? Well, we can view $\mathcal{L}$ as a path from one endpoint to the other, and describe that path. Better yet, all we have to do is to describe the relationship between the two endpoints; once we do that, it's an easy task to connect them with a straight line segment. For example, the relationship between the endpoints of $\mathcal{L}$ can be described as follows:

Move 2 units to the right and 1 unit up.

Note that this instruction *does not depend on where we start*: pick any point A, follow the instructions above to arrive at a second point B, and now connect A to B, and you have drawn $\mathcal{L}$!

Because it's tiresome to write everything in words, we introduce a couple of symbols. Let $\vec{e}_1$ denote '*move one unit to the right*' and $\vec{e}_2$ denote '*move one unit up*'. (The arrows are there to remind you that these variables aren't numbers or points, but instead are a direction of motion.) Using this notation, we can write $\mathcal{L}$ more succinctly as

$$\mathcal{L} = 2\vec{e}_1 + \vec{e}_2.$$

Similarly,

$$f(\mathcal{L}) = \vec{e}_1 + 3\vec{e}_2.$$

Or to put it all in a single equation:

$$f(2\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + 3\vec{e}_2. \tag{$\dagger$}$$

Now, we haven't actually proved that this is true yet – all we did was to verify this in the two cases of starting point $(0, 0)$ and starting point $(-1, -1)$. By contrast, (†) asserts a relationship which holds *no matter which starting point you choose*. We will pick this up next time.