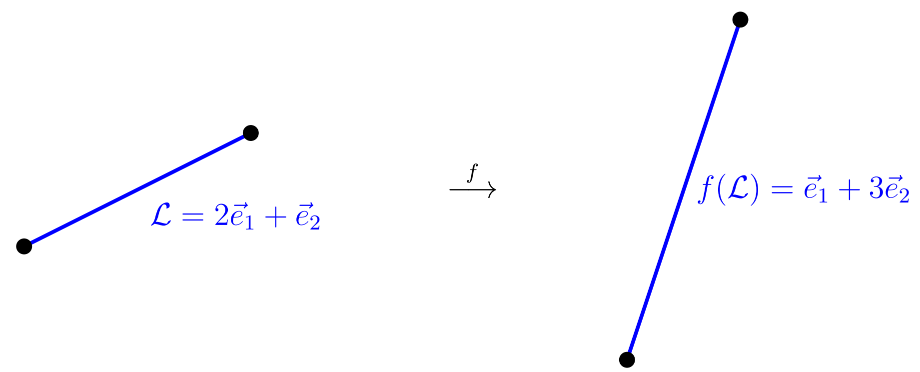
LINEAR ALGEBRA: LECTURE 15

LEO GOLDMAKHER

Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) := (x - y, 2x - y).$$

Last time we noticed that if we apply this map to the line segment $\mathcal{L}$ pictured below, the appearance of the image $f(\mathcal{L})$ depends only on the appearance of $\mathcal{L}$, and is *independent* of the location of $\mathcal{L}$. Here's an illustration of the situation:



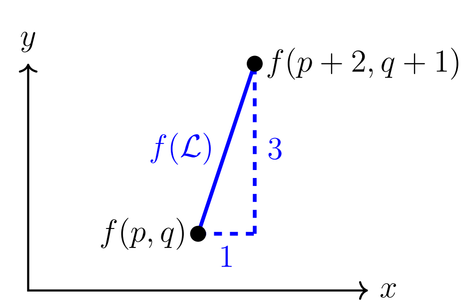
Actually, we never proved this to be true – we merely observed it for two specific realizations of $\mathcal{L}$. However, it's not so difficult to prove in general. Let's try it:

Claim. $f(2\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + 3\vec{e}_2$

Proof. Pick an arbitrary point $(p, q) \in \mathbb{R}^2$, and let $\mathcal{L}$ denote the line segment corresponding to the vector $2\vec{e}_1+\vec{e}_2$ starting at (p, q). In other words, $\mathcal{L}$ is the segment connecting (p, q) to $(p + 2, q + 1)$. From problem **4.7** we know that the image of this line, $f(\mathcal{L})$, is itself a line segment connecting the points $f(p, q)$ and $f(p+2, q+1)$. Since f is additive, we have

$$f(p + 2, q + 1) = f(p, q) + f(2, 1) = f(p, q) + (1, 3)$$

Here's an illustration of the image of $\mathcal{L}$:

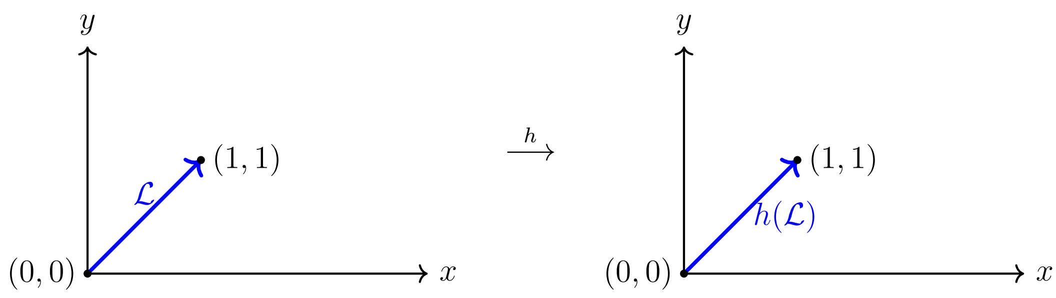


Of course, the location of $f(\mathcal{L})$ very much depends on the choice of (p, q). *But the shape of* $f(\mathcal{L})$ *does not depend on* (p, q): it simply corresponds to the vector $\vec{e}_1 + 3\vec{e}_2$. Thus, no matter where in the plane we place the vector $2\vec{e}_1 + \vec{e}_2$, its image under f is the vector $\vec{e}_1 + 3\vec{e}_2$. In short: $f(2\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + 3\vec{e}_2$. $\square$

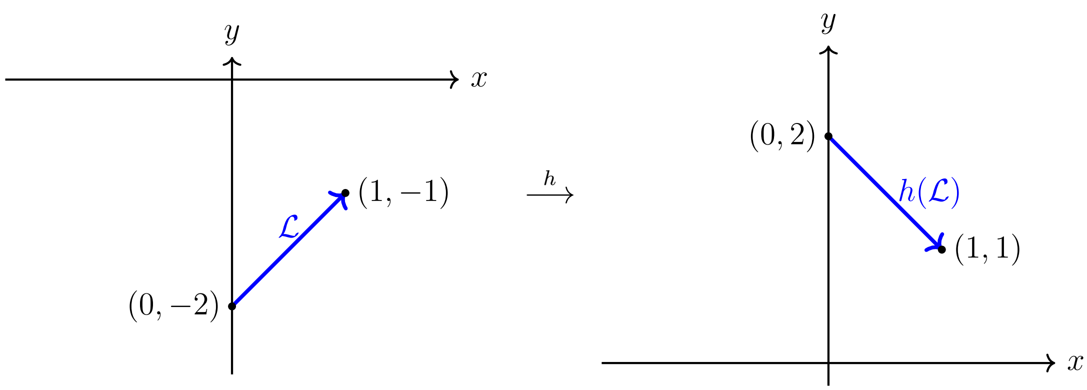
Date: March 11, 2016.

Relationships like this claim may not strike you as surprising – the function f maps points to points and line segments to line segments, so of course it should map one set of directions to another set of directions. The rub is that most functions aren't well-defined on any particular set of directions.

Nonexample. Consider the function $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x, y) := (x, |y|)$. What is $h(\vec{e}_1 + \vec{e}_2)$? To answer this question, we draw $\vec{e}_1 + \vec{e}_2$, and then pick a coordinate system. Once we do this, we can label the two endpoints and see where they get sent by h. Finally, we can rewrite directions from one endpoint to the other in the language of $\vec{e}_1$ and $\vec{e}_2$. Here we go:



Thus, one might be tempted to conclude that $h(\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + \vec{e}_2$. However, this is not the case. For example, consider the following picture:



In this case, we have $h(\vec{e}_1 + \vec{e}_2) = \vec{e}_1 - \vec{e}_2$, which contradicts our above result. In other words, we conclude that $h(\vec{e}_1 + \vec{e}_2)$ is not well-defined! To emphasize: h is perfectly well-defined at any particular point, but on the level of directions (i.e., vectors), it is no longer well-defined.

The notion of well-defined is subtle, so we gave one more example of a function which isn't well-defined.

Nonexample. Consider the following:

$$\begin{aligned} g : \mathbb{Q} &\longrightarrow \mathbb{Z} \\ \frac{a}{b} &\longrightarrow a + b \end{aligned}$$

At first glance, this looks like a perfectly reasonable function. But a bit more thought shows that it's not well-defined. For example, what's $g(1/3)$? Well, it's $1+3 = 4$. But also, since $\frac{1}{3} = \frac{2}{6}$, we have $g(1/3) = g(2/6) = 8$. This contradiction shows that g is not well-defined.

We now arrive at the main result for the day. To state it succinctly, we introduce one more piece of notation: let

$$V^2 := \{\alpha\vec{e}_1 + \beta\vec{e}_2 : \alpha, \beta \in \mathbb{R}\}$$

denote the set of all vectors (in the plane). We prove:

Proposition 1. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then it is well-defined when viewed as a function from $V^2 \to V^2$.*

Proof. We must show that the output of f does not depend on which coordinate system we pick. Let $\alpha\vec{e}_1 + \beta\vec{e}_2$ be an arbitrary vector. It will be helpful to label the coordinates of the point $f(\alpha, \beta)$; call it (h, k).

Now suppose we set up a coordinate system somehow, and that the vector $\alpha\vec{e}_1 + \beta\vec{e}_2$ starts at some point (p, q). Then it ends at $(p + \alpha, q + \beta)$. Applying f yields a line segment from $f(p, q)$ to $f(p + \alpha, q + \beta)$. Note that by additivity,

$$f(p + \alpha, q + \beta) = f(p, q) + f(\alpha, \beta) = f(p, q) + (h, k).$$

Ignoring the coordinates we set up, we deduce that the output vector is $h\vec{e}_1 + k\vec{e}_2$. In short, we've shown that

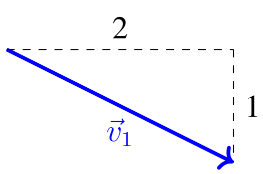
$$f(\alpha\vec{e}_1 + \beta\vec{e}_2) = h\vec{e}_1 + k\vec{e}_2, \tag{$\dagger$}$$

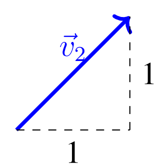
where h and k are defined by $(h, k) := f(\alpha, \beta)$. But now comes the key observation: nothing appearing in (†) depends on (p, q)! In other words, the output vector of $f(\alpha\vec{e}_1 + \beta\vec{e}_2)$ depends only on the input vector, and not on which coordinate system we picked to view it in. This concludes the proof. □

In the final few minutes of the lecture, we introduced the notion of change of basis. All of our descriptions of vectors thus far have been in terms of two basic directions:

$\vec{e}_1$: Move one unit to the right.
$\vec{e}_2$: Move one unit up.

Now suppose an alien pays us a visit. On its planet, the two basic directions are $\vec{v}_1 := 2\vec{e}_1 - \vec{e}_2$ and $\vec{v}_2 := \vec{e}_1 + \vec{e}_2$, pictured below:





We want to be nice hosts, and to explain to the alien the location of a delicious local restaurant: you just have to walk three blocks east and two blocks north. How does one explain this to the alien, when the alien doesn't know what *north* and *east* mean? One nice strategy is to express $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\vec{v}_1$ and $\vec{v}_2$. This isn't so hard to do. For example, we know that walking $\vec{v}_1$ followed by $\vec{v}_2$ is the same as walking two blocks east, then one block south, then one more block east, then one block north. An easier way to say this is: walk three blocks east. Or in symbols:

$$\vec{v}_1 + \vec{v}_2 = 3\vec{e}_1.$$

This gives us

$$\vec{e}_1 = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2.$$

Similarly,

$$\vec{e}_2 = -\frac{1}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2.$$

Now we can give directions to the restaurant easily:

$$\begin{aligned} 3\vec{e}_1 + 2\vec{e}_2 &= \vec{v}_1 + \vec{v}_2 - \frac{2}{3}\vec{v}_1 + \frac{4}{3}\vec{v}_2 \\ &= \frac{1}{3}\vec{v}_1 + \frac{7}{3}\vec{v}_2. \end{aligned}$$

We'll discuss these ideas in a bit more depth and generality next time.